BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 1, January 1994

0273-0979/94 $1.00 + $.25 per page

Geometry and spectra of compact Riemann surfaces, by P. Buser. Progress in Mathematics, vol. 106, Birkhäuser, Boston, 1992, xiv + 454 pp., $69.50. ISBN 0-8176-3406-1.

A Riemann surface is by definition a complex manifold of dimension 1; it can be regarded as having a conformal structure. This means that one can measure angles on it between any two tangent vectors at a given point. Essentially a holomorphic function is one that preserves angles (and orientation), so, as was stressed by Riemann himself, this concept can serve as the basis for the study of complex analysis and for the theory of Riemann surfaces. It is possible to prove that there is a complete Riemannian metric giving rise to this conformal structure and which is "as regular as possible" in that it has constant curvature. The cases where the curvature is positive or zero can be characterized as being the Riemann sphere, the plane, the plane with one puncture, or a torus (elliptic curve). All the other cases have negative curvature. When we restrict our attention to this latter class, we can rescale the metric so that it has negative curvature of -1. This not only introduces the notion of length to a Riemann surface but also gives an absolute unit for it, just as the curvature of the earth gave rise to the first definition of the meter. The universal covering surface will then be the standard simply connected two-dimensional surface of constant negative curvature, for which there are several models; and, if we like, we can choose one according to our personal preference.

There are two questions that present themselves and that are the subject matter of this book. The first consists of the existence and classification of Riemann surfaces. The other concerns the analogue of the Laplace operator on a (compact) Riemann surface when we have agreed to make use of the "best" metric available, namely, the one above. (There are others, such as the Bergmann metric, which are in some ways rather more natural, but we shall not go into that here.) This analogue is called the Laplace-Beltrami operator. The two problems come together in Gelfand's isospectral problem, of which we shall say more later.

The question of the existence and classification of Riemann surfaces has a very long history, which we shall not repeat here, and several deep and interesting answers. One particularly attractive and quite elementary approach, due apparently to Lipman Bers, is through the decomposition of a compact Riemann surface into basic pieces usually called "pairs of pants", although Buser prefers the more decorous phrase "Y-pieces". These surfaces are glued together, and the geometric parameters of this process essentially classify Riemann surfaces up to conformal equivalence. This construction is one of the two major themes in this book.

The other major theme is the spectral theory of the Laplace-Beltrami operator. It follows from general principles that the spectrum is discrete, and, thanks to Weyl's law, one knows its asymptotic distribution. It is, however, very difficult to go further than these general principles for arbitrary Riemannian manifolds or, for that matter, for the very analogous case of Schrödinger operators where there is a strongly attractive potential. Riemann surfaces provide a test case in which one can both formulate the problems more precisely and investigate them more deeply than in the general case. Moreover, as was discovered in the 1940s and 1950s by Delsarte, Maaß, Huber, and Selberg, there is a very strong connection between the spectral theory of the Laplace-Beltrami operator and harmonic analysis on $\mathrm{PSL}(2, \mathbb{R})$. This finds its ultimate expression in the Selberg trace formula, which can also be thought of as an expression of Bohr's correspondence principle in quantum mechanics—here between the spectral (i.e., quantum) theory of the Laplace-Beltrami operator and its "classical" analogue, the geodesic flow. The Selberg trace formula and the penultimate step in the proof, the pretrace formula, mean that there is a very strong connection between the geometry of a Riemann surface, in particular the lengths of closed geodesics, and the spectral theory of the Laplace-Beltrami operator. There is another connection between the geometry of the Riemann surface and the Laplace-Beltrami operator. When we study small eigenvalues of an elliptic differential operator, one of the most effective methods is the use of variational principles to characterize the eigenfunctions and eigenvalues. This is an extension of Dirichlet's principle which Riemann stressed as being at the basis of the function theory of Riemann surfaces. The applications of this method result in converting problems about eigenfunctions into problems about geometry, which one then has to solve as best one can.

One of the questions that has stimulated the most research is what has gone into the literature under the name "Gelfand's conjecture". This appears in a paper Gelfand and Piatetski-Shapiro wrote in 1959 (*Representation theory and theory of automorphic functions*, Uspekhi Mat. Nauk **14** (1959), no. 2, 171–194; English transl. Trans. Amer. Math. Soc. (2) **26** (1963), 173–200) and asks whether the set of eigenvalues ("eigenvalue spectrum") of the Laplace-Beltrami operator serves to characterize the Riemann surface, just as in M. Kac's famous "Can one hear the shape of a drum?" paper of 1966. Using the trace formula, Gelfand and Piatetski-Shapiro proved that the eigenvalue spectrum determined the "length spectrum", that is, the set of lengths of closed geodesics (but not which geodesics had these lengths) and could show that a continuous deformation of the Riemann surface which preserves the length spectrum is trivial. They posed the question as to whether the length spectrum or the eigenvalue spectrum determines the Riemann surface uniquely. H. P. McKean showed that the set of Riemann surfaces with the same spectrum was finite; Buser shows here that "finite" means $\leq \exp(720\, g^2)$, where g is the genus of the Riemann surface. A negative answer was first given by M.-F. Vignéras, and more recently T. Sunada has shown how to construct many examples. One of the high points in this book is the chapter on Wolpert's theorem, which sharpens the theorem of Gelfand and Piatetski-Shapiro by showing that the length spectrum really does determine the spectrum, except for an exceptional set of Riemann surfaces of lower dimension. Another remarkable result along the same line of thought is proved in the last chapter, namely, that a finite part of the eigenvalue spectrum determines the whole spectrum. In view of what has gone before, this is not perhaps surprising, but it is remarkable that one can give almost explicit bounds.

Although the examples of the spectra of the Laplace-Beltrami operator on compact Riemann surfaces are of considerable value in understanding more general situations, the real excuse for studying them is the stimulus these questions give to geometrical studies of Riemann surfaces, and that is the central theme here. This is a thick and leisurely book which will repay repeated study with many pleasant hours—both for the beginner and the expert. It is fortunately more or less self-contained, which makes it easy to read, and it leads one from essential mathematics to the "state of the art" in the theory of the Laplace-Beltrami operator on compact Riemann surfaces. Although it is not encyclopedic, it is so rich in information and ideas that, rather than complaining about what is missing, the reader will be grateful for what has been included in this very satisfying book.

S. J. PATTERSON
MATHEMATISCHES INSTITUT DER GEORG-AUGUST-UNIVERSITAT
E-mail address: spatter@gwdgv1.dnet.gwdg.de

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AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 1, January 1994

0273-0979/94 \$1.00 + \$.25 per page

Differential manifolds, by Antoni A. Kosinski. Academic Press, New York, 1992, xvi + 248 pp. ISBN 0-12-421850-4.

This is a book that deals with an exciting ten-year period—roughly 1954–1964—of epochal discoveries in the theory of differential (or differentiable) manifolds and is, to the best of my knowledge, the first systematic, comprehensive account of them written as an upper-level graduate text or for independent study. Ten years is a rather short time in the rich history of this subject, and it may be useful to give some idea of how the core subjects covered in the book fit in.

First, some definitions. A (topological) *manifold* M of dimension n is a locally Euclidean Hausdorff space satisfying some separability condition, say, with a countable basis of open sets. *Locally Euclidean* means that each point in M lies in a coordinate neighborhood, an open set U equipped with a homeomorphism ϕ to an open subset of Euclidean space R^n. It becomes a *differential manifold* if it can be covered with a collection of such neighborhoods $\{U_\alpha, \phi_\alpha\}$ such that the change of coordinates $\phi_\beta \circ \phi_\beta^{-1}$ is smooth (i.e., C^∞) whenever U_α, U_β overlap. Any such collection is called a *differential structure* on M. A manifold M is *triangulable* if it can be decomposed into simplices (or other simple polyhedra, e.g., cubes) fitting nicely along their faces. Each of these three definitions carries a corresponding notion of equivalence, namely, homeomorphism, diffeomorphism (differentiable homeomorphism), and piecewise linear equivalence, respectively. It is the differential structure which allows the use of calculus on manifolds, differentiable functions and mappings, vector fields, tensors, and so on.

Although these definitions are now standard and known to most mathematicians, they took some eighty years, beginning with the work of Riemann until Whitney's 1936 paper [11], to achieve this precise form. It is hardly an exaggeration to say that manifolds are the most important spaces in mathematics. In