tions give to geometrical studies of Riemann surfaces, and that is the central theme here. This is a thick and leisurely book which will repay repeated study with many pleasant hours—both for the beginner and the expert. It is fortunately more or less self-contained, which makes it easy to read, and it leads one from essential mathematics to the "state of the art" in the theory of the Laplace-Beltrami operator on compact Riemann surfaces. Although it is not encyclopedic, it is so rich in information and ideas that, rather than complaining about what is missing, the reader will be grateful for what has been included in this very satisfying book.

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Differential manifolds, by Antoni A. Kosinski. Academic Press, New York, 1992, xvi + 248 pp. ISBN 0-12-421850-4.

This is a book that deals with an exciting ten-year period—roughly 1954–1964—of epochal discoveries in the theory of differential (or differentiable) manifolds and is, to the best of my knowledge, the first systematic, comprehensive account of them written as an upper-level graduate text or for independent study. Ten years is a rather short time in the rich history of this subject, and it may be useful to give some idea of how the core subjects covered in the book fit in.

First, some definitions. A (topological) *manifold* M of dimension n is a locally Euclidean Hausdorff space satisfying some separability condition, say, with a countable basis of open sets. *Locally Euclidean* means that each point in M lies in a coordinate neighborhood, an open set U equipped with a homeomorphism ϕ to an open subset of Euclidean space R^n. It becomes a *differential manifold* if it can be covered with a collection of such neighborhoods $\{U_\alpha, \phi_\alpha\}$ such that the change of coordinates $\phi_\beta \circ \phi_\beta^{-1}$ is smooth (i.e., C^∞) whenever U_α, U_β overlap. Any such collection is called a *differential structure* on M. A manifold M is *triangulable* if it can be decomposed into simplices (or other simple polyhedra, e.g., cubes) fitting nicely along their faces. Each of these three definitions carries a corresponding notion of equivalence, namely, homeomorphism, diffeomorphism (differentiable homeomorphism), and piecewise linear equivalence, respectively. It is the differential structure which allows the use of calculus on manifolds, differentiable functions and mappings, vector fields, tensors, and so on.

Although these definitions are now standard and known to most mathematicians, they took some eighty years, beginning with the work of Riemann until Whitney's 1936 paper [11], to achieve this precise form. It is hardly an exaggeration to say that manifolds are the most important spaces in mathematics. In addition to the Euclidean spaces R^n, the unit spheres S^{n-1} in R^n, and other standard examples, manifolds include the spaces of classical geometries (e.g., real or complex projective space), Riemann surfaces, Lie groups, symmetric spaces, phase spaces of dynamical systems, nonsingular algebraic varieties, and others. They are important almost everywhere in mathematics from calculus on.

As soon as the definitions are given, problems come to mind, the most immediate being to construct systematically examples in each dimension and, if one is an optimist, to look for some classifying properties. How many such spaces are there? The topological conditions imply that the connected components, themselves manifolds, are open and closed and countable in number. When $n = 1$, the only connected manifolds are the real line R and circle S^1. The case $n = 2$, or surfaces, is more interesting: all compact, connected, orientable manifolds may be obtained by the familiar procedure of removing pairs of open disks from S^2 and pasting a cylinder, or handle, by fastening each of its end circles to the boundary circle of a removed disk. A variation of this procedure yields the nonorientable ones as well. These manifolds have been known for more than a century, albeit without rigorous proofs until about 1925. (We note in passing that the disk and cylinder mentioned here, in addition to line segments and rays, are examples of a slightly more general concept: manifold M *with boundary* ∂M, a manifold of dimension one less. To include these, the definitions need only be slightly modified: allow the coordinate neighborhoods U, f to include homeomorphisms into the half space $R^n_+\{(x_1, \dots, x_n): x_n \geq 0\}$. The points with coordinates for which $x_n = 0$ constitute ∂M, a manifold of dimension $n - 1$.)

Beyond classification and questions of examples, many of which may be constructed by pasting together polyhedra or manifolds with boundary in various ways, each of the three definitions above raises fundamental questions. For example, does every manifold admit a differentiable structure; and if so, is it unique to within diffeomorphism? Is every manifold triangulable; and if so, is this structure unique (the Hauptvermutung) to within piecewise linear equivalence? Does triangulability imply a differential structure or vice versa? The fact that these questions were answered fairly early for $n = 2$ but proved to be a very elusive and long-resisted solution for higher dimensions probably contributed to the development of this subject.

Both manifold theory and algebraic topology owe everything to the insight, originality, and genius of Henri Poincaré, who, beginning around 1892 until his death in 1912, returned to this area again and again, introducing strikingly original ideas and crucially important tools and raising basic questions that are still studied today. In particular, we owe to him the idea of associating groups to manifolds, groups *invariant* with respect to homeomorphisms, and thus measuring topological properties of these spaces such as genus and connectivity, which were previously measured by numbers. He did this in two ways: first, he introduced the homology groups, which already proved sufficient to classify the compact two-dimensional manifolds; and second, he defined the fundamental group $\pi_1(M, p)$ whose elements are equivalence classes of loops in M from $p \in M$ with respect to continuous deformation of the loop. The first led to homology and cohomology theory, and the second ultimately resulted in the use by Hopf and Hurewicz of homotopy, homotopy groups $\pi_n(M, p)$, and a new equivalence—homotopy equivalence. The elements of $\pi_n(M, p)$ are deformation classes of continuous mappings of S^n into M taking a fixed point of S^n to p. (Note that a loop at p is a continuous image of S^1 passing through p.) Two spaces X and Y are *homotopically equivalent* if there are continuous mappings $f: X \to Y$ and $g: Y \to X$ whose compositions $f \circ g$ and $g \circ f$ can be continuously deformed to the identity maps of X and Y; homotopy groups are the same for X and Y. One of the most famous mathematical conjectures is related to these considerations. S^1 and S^2 are easily characterized as compact orientable manifolds by the groups introduced by Poincaré, but he was unable to so characterize S^3. He then asked if any compact 3-dimensional manifold for which π_1 is trivial is homeomorphic to S^3. All attempts to prove this and its generalization to S^n, $n > 3$, failed for more than fifty years. These efforts resulted in one of the best-known unsolved problems in mathematics, which became known as the (generalized) Poincaré Conjecture: If a compact orientable (differential) manifold is homotopically equivalent to S^n, then is it homeomorphic to S^n? M is homotopically equivalent to S^n if $\pi_1(M, p)$ is trivial and it has the same homology groups as S^n or, again, if every continuous map $f: S^k \to M$ is deformable to a mapping to a single point p for all $k < \dim M$.

Poincaré's work, like that of Riemann and other predecessors, was very geometric and intuitive, without precise proofs, and often difficult to follow (see the comments of Dieudonné [2]). Careful, rigorous methods were developed by Weyl for Riemann Surfaces (for some interesting history see [12]), Brouwer for simplicial complexes (thus triangulated manifolds), and others. Now on a firm foundation, homology theory and later homotopy theory developed enormously with many successes in the following years but often in topological spaces much more general than manifolds, leaving the fundamental questions involving them in the background, impenetrable and seemingly unknowable.

Even though Poincaré's manifolds were both piecewise linear and differentiable, it was not known how to use this additional structure effectively in answering some of the fundamental questions about manifolds. One notable exception as it now appears was the work of Marston Morse in the 1920s that relates the critical points of a function on a differential manifold to its homology. Then in 1936 Whitney in a famous paper [11] gave a careful definition of smooth manifolds (the one above) and showed that they were all diffeomorphic to submanifolds of $M \subset R^N$, which had been assumed without proof by almost all the pioneers from Riemann on. This coincided with the introduction of the concept of fiber bundle developed by Whitney and others. The most natural examples are the "bundle" or space of all vectors tangent to $M \subset R^N$ and the bundle of all vectors normal to it. These are themselves differential manifolds with obvious smooth projections onto M taking each vector to its initial point. From this point on, "differential topology" came into its own.

A decisive event in our story was the creation of cobordism theory in the early fifties by Thom, which shortly thereafter (1956) was used by Milnor in a short paper [7] in which he showed that there are differential structures on the 7-sphere S^7 which are not equivalent under any diffeomorphism. This profound and unexpected discovery was followed by a veritable explosion of results; and it is these, all occurring within ten years or less, which form the subject of the book being reviewed. Within a very short time it was shown that there were, in fact, twenty-eight differential structures on the oriented S^7 and that they even formed a group! A fundamental paper of Milnor and Kervaire [5] reduced the problem to homotopy groups of spheres and settled it for many dimensions. Smale [8] proved the generalized Poincaré conjecture in the differentiable case for dimensions > 4 (i.e., a smooth manifold M^n whose homotopy type is that of S^n is diffeomorphic to S^n) and developed his handlebody theory [9], a particularly elegant way to build all differentiable manifolds by a generalization of the pasting technique described above for compact surfaces. The breakthroughs in understanding differential manifolds led to corresponding results in other fundamental questions mentioned above. It was shown that there are manifolds with no differentiable structure, manifolds with no triangulation, manifolds with inequivalent triangulations (counterexample to the Hauptvermutung), that the Poincaré conjecture in dimension 4 was proved for topological manifolds, and more. For a precise picture see the historical notes and references in the book being reviewed.

In the intervening years since these events there have been books presenting various aspects of differential topology; [1, 3, 4, 6, 10] come to mind, and there are no doubt others not known to the reviewer. Hirsch's book [4] is closest to the spirit of the present book and is a very good introduction to it. Now that some 25–30 years have elapsed since these discoveries were made, some of the most difficult concepts have been clarified, extended, and refined, and the proofs simplified so that happily the subject is now mature enough for a simplified and unified treatment. The book being reviewed begins from a basic knowledge of differentiable manifolds and algebraic topology (including some homotopy theory) and proceeds to give a quite complete account with carefully written demonstrations of the central core of these great theorems. Except for the algebraic topology, of which more than a passing knowledge is needed, it is relatively self-contained. The first eighty pages develop the elementary manifold theory and differential topology needed for the sequel, more or less from the definition of differential manifold. In particular, basic ideas and results on transversality, isotopies, tubular neighborhoods and collars, and some Morse Theory are included. These are topics which would normally not be familiar to someone with just a basic knowledge of smooth manifolds, although they are readily available in the earlier books mentioned above. Then, following a short chapter on foliations that is not used elsewhere in the book, the author begins a systematic development of material that is not, to the reviewer's knowledge, available in other texts. There is a detailed discussion of operations on manifolds, i.e., cutting and pasting of various types such as: connected sum, attaching handles, surgery, and so on. This is very important and contains much of the geometric intuition; it describes the way in which manifolds are put together from known manifolds and generalizes the familiar picture mentioned before of fastening cylinders to the sphere to build up surfaces. These preliminaries take up a little more than half of the book.

The final four chapters contain the central core of the book. In them the author presents and proves in detail the handlebody theorems, develops cobordism, h-cobordism, and framed cobordism theories, and then discusses surgery. In the course of doing so he proves many of the famous theorems of Thom, Milnor, Smale, and others dealing with differential manifolds, including the h-

cobordism theorem, the determination of the groups of differential structures on spheres, the handlebody presentation theorem, and the proof of the differential Poincaré conjecture for $n > 4$. He has done an excellent job of organizing this deep and difficult material so that it can be presented in a relatively short book. The material is broken up so that there are virtually no long proofs; the style is lively, and the writing is clear. For the independent reader I would have preferred a slightly more leisurely pace, more intuitive discussion, and more illustrations in some places, particularly the chapter on operations on manifolds; but when it is used as a text, such commentary can be supplied by the instructor. As already noted, the prospective reader should be aware that although the basics of differential manifolds are carefully, if compactly, presented, there is no preparation for the homology and homotopy theory that is needed. On the other hand, there is an excellent bibliography with references to expository material, fundamental research papers, and even historical papers; so there is no problem in finding sources to fill in any gaps. There are, in fact, many interesting historical notes, which are much appreciated by this reviewer. Manifold theory has been an important, central part of mathematics for a century; and all readers, students most particularly, will be edified and inspired by knowing the history of this beautiful and important subject.

It should add to the attractiveness of this book that the story is far from complete and is an extremely active subject of research now and in the recent past. Many of the results above are for dimensions > 4, leaving important questions in dimensions 3 and 4 unsettled. For example, it was discovered in the 1980s that familiar Euclidean space R^4 has inequivalent differential structures! Moreover, amazingly, these seemingly arcane and exotic ideas have suddenly and unexpectedly come to be important in theoretical physics. Finally, it should not be forgotten that some of the original questions, in particular, the Poincaré Conjecture in dimension 3, appear to be still unsolved; nor is it known if S^4 possesses inequivalent differential structures. How useful for mathematics it is to have a single, short, well-written book on differential topology where one can study the great discoveries of the sixties and prepare for exciting new developments of the nineties.

References

1. Th. Brocker and K. Janich, *Introduction to differential topology*, Cambridge Univ. Press, Cambridge, 1982.
2. J. Dieudonné, *A history of algebraic and differential topology* 1900–1960, Birkhäuser, Boston, MA, 1989.
3. V. Guillemin and A. Pollack, *Differential topology*, Prentice Hall, Engelwood Cliffs, NJ, 1974.
4. M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976.
5. M. Kervaire and J. Milnor, *Groups of homotopy spheres*, Ann. of Math. (2) **77** (1963), 504–537.
6. J. Milnor, *Morse theory*, Princeton Univ. Press, Princeton, NJ, 1963.
7. ———, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. (2) **64** (1956), 399–405.
8. S. Smale, *The generalized Poincaré conjecture in higher dimensions*, Bull. Amer. Math. Soc. (N.S.) **66** (1960), 387–399.

10. A. Wallace, *Differential topology*, W. A. Benjamin, New York, 1968.

11. H. Whitney, *Differentiable manifolds*, Ann. of Math. (2) **37** (1936), 645–680.

12. A. Shields, *Years ago*, Math. Intelligencer **10** (1988), 5–8.

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Systems of evolution equations with periodic and quasiperiodic coefficients, by Yu. A. Mitropolsky, A. M. Samoilenko, and D. Martinyuk. Mathematics and Its Applications (red series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993, xiv+280 pp., \$126.00. ISBN 0-7923-2054-9

To unify somehow the content of the book, the authors have chosen the key words "Evolution Equations". It is not my intent to propose another key word, but I have the feeling that the real unifying element is the flavor of the Kiev School on Nonlinear Oscillations, initiated about six decades ago by Krylov and Bogoliubov. Let us mention some of the main publications that are characteristic of the development of this school. It started with Nicolai Mitrofanovich Krylov (born in 1879 and educated in St. Petersburg but working in Kiev since 1922). If one looks at the list of papers published by Krylov and by Bogoliubov and Krylov, one finds some of the same themes that are present in the book under review, such as "Approximation of periodic solutions of differential equations", a paper in French published in 1929 by Krylov, or "On the quasiperiodic solutions of the equations of the nonlinear mechanics", published in French in 1934 by Krylov and Bogoliubov.

The most famous publication of Krylov and Bogoliubov is their book *Introduction to nonlinear mechanics*, published in Kiev in 1937. What was typical of all these pioneering works of the Kiev School was the stress on computational aspects, on motivations, and on applications. In the sequel the main theoretical aspects were brilliantly clarified by N. N. Bogoliubov in a rather unknown monograph entitled *On certain statistical methods in mathematical physics* (Kiev) published in 1945; it was there the "method of averaging" found a full mathematical justification and the idea of reducing the problem by considering integral manifolds was pointed out.

The next event was the widely known book by Bogoliubov and Mitropolskii, *Asymptotic methods in the theory of nonlinear oscillations*, published in Russian in 1955, followed by a long series of books by Mitropolskii and his colleagues; among them let us mention that by Mitropolskii and Lykova, *Integral manifolds in nonlinear mechanics* (Nauka, Moscow, 1974). After discovering that the main ideas of the "accelerated convergence" procedure of Kolmogorov, Arnold, and Moser were already present in the work of Bogoliubov in the 1969 monograph *The methods of rapid convergence in nonlinear mechanics* (Naukova Dumka, Kiev), Yu. A. Mitropolskii and A. M. Samoilenko gave the Kiev version of this approach. Under the direction of Mitropolskii in Kiev a large number of studies were performed, dedicated mostly to the extension of the methods to different classes of equations and to various applications.