# BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 1, January 1994

0273-0979/94 $1.00 + $.25 per page

*Elie Cartan* (1869–1951), by M. A. Akivis and B. Rosenfeld, translated by V. V. Goldberg, 1991. American Mathematical Society, Providence, RI, 1993, xii+318 pp., $123.00. ISBN 0-8218-4587-X

This is a carefully written and rather complete biography of one of the greatest mathematicians of the twentieth century.

Cartan's work covers all of differential geometry in the broad sense, from Lie algebra and pseudogroups to differential systems and topology. Many of his ideas and results are fundamental. I will single out a few that are particularly noteworthy.

The first is his exterior differential calculus. The standard analytic tool in Riemannian geometry is Ricci's tensor calculus, which is both elegant and effective. Comparing the two, I would call the latter a secondary operation whose introduction needs a connection on the manifold, while the former is a primary operation depending only on the differentiable structure. This distinction has far-reaching consequences, for in the last analysis a primary operation is more significant than a secondary one. Most geometers, including the great authority, Hermann Weyl, found Cartan's work hard to follow. I suspect that one reason lies in their training through tensor analysis, which becomes a burden.

Cartan introduced the exterior differential calculus in his book on "invariants intégraux" (1992), following Frobenius's treatment of the Pfaffian form and its bilinear covariant. The effects are remarkable, and I mention the following:

1. The transformations of a Lie group can be regarded as those leaving the Maruer-Cartan forms invariant. This can be generalized to the infinite pseudogroups of Cartan.

2. A $G$-structure on a manifold is a set of Pfaffian forms defined up to a linear group $G$ in the tangent bundle of the manifold. The most notable example is when $G$ is the orthogonal group and the resulting $G$ structure is then the Riemannian structure. By introducing auxiliary variables, we have locally in $M \times G$ invariant Pfaffian forms. The exterior differential calculus gives a solution of the equivalence problem. In the Riemannian case it leads quickly to the Levi-Civita parallelism and gives a solution of the form problem. More generally, in many classical cases the solution of the equivalence problem introduces a connection. It is described analytically by a linear differential form (=Pfaffian form). By exterior differentiation its curvature is described by a quadratic exterior differential form. These results fit well with the global

theory of fiber bundles and lead naturally to the curvature definition of the characteristic classes.

3. Exterior differential systems include any system of partial differential equations. The theory is intrinsic and adapts well to nonlinear equations.

This list could continue. Exterior differential calculus is destined to occupy a more important place in multivariable calculus!

Of which works was Cartan most proud? Through my conversations with him during my Paris séjour of 1936–1937, I would like to venture a guess. I would suggest that it is his works on linear representations, including his discovery of spinors in 1913. He published a book on spinors in 1938 in which he included the physical applications.

In the conclusion of the book the authors state: "As a rule, Cartan built his scientific research on works of his predecessors, developing their ideas so well that other mathematicians often forgot the original works." This clearly does not apply to the works mentioned above. Moreover, in the examples used by the authors to illustrate their statement—moving frames and generalized spaces—Cartan developed the methods and ideas for homogeneous spaces with any Lie group, which went far beyond the scope of his predecessors.

Cartan roamed through a vast and fertile area of mathematics. With his power and insight he was able to pick up the gems wherever he treaded. His books are full of interesting details. This was not the case with his earlier works, such as pseudogroups and exterior differential systems, which were original but needed clarification. As a result the recognition of his achievements came late. It was perhaps Hermann Weyl's work on group representations in 1925–1926 that made Cartan famous in the general mathematical community. Although Poincaré had a high opinion of the role of a "group" in mathematics and of Cartan's contribution to it (his report on Cartan's work is included in the book), Cartan was elected a member of the French Academy only in 1931.[1] I have no doubt that Cartan realized the importance of his works. He was able to ignore the outside reaction and led a simple, happy, and fruitful life.

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*Elliptic curves*, by Anthony W. Knapp. Princeton University Press, Princeton, NJ, 1992, xv+427 pp., $24.95. ISBN 0-691-08559-5

This book is about elliptic curves and modular functions, two topics that are intimately related in both accidental and essential ways. As emphasized by André Weil in his magisterial historical introduction to contemporary number theory [W], the arithmetic study of elliptic curves is, in spite of the clear

[1]This could be a result of the outdated French system. The number of members in a section was fixed, so that a new member could be elected only when an old one died. To my knowledge this system has changed.