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MÜNTZ SPACES AND REMEZ INEQUALITIES

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ABSTRACT. Two relatively long-standing conjectures concerning Müntz polynomials are resolved. The central tool is a bounded Remez type inequality for non-dense Müntz spaces.

1. INTRODUCTION

Müntz's beautiful, classical theorem characterizes sequences $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.1}$$

for which the Müntz space $M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0, 1]$. Here, and in what follows, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients and $C[A]$ is the space of all real-valued continuous functions on $A \subset [0, \infty)$ equipped with the uniform norm. Throughout this paper $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ denotes a sequence satisfying (1.1). Müntz's Theorem [9, 11, 17, 24, 27] states the following.

Theorem. *$M(\Lambda)$ is dense in $C[0, 1]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.*

The original Müntz Theorem proved by Müntz [17] in 1914 and by Szász [24] in 1916 and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0, 1]$ by an interval $[a, b] \subset [0, \infty)$ in Müntz's Theorem is a nontrivial issue. This is, in large measure, due to Clarkson and Erdős [12] and Schwartz [22] whose works include the result that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then every function belonging to the uniform closure of $M(\Lambda)$ on $[a, b]$ can be extended analytically throughout the region $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$.

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There are many variations and generalizations of Müntz's Theorem [1, 4, 5, 6, 7, 8, 9, 16, 18, 22, 23, 25, 26]. There are also still many open problems.

In Section 3 of this paper we show that the interval $[0, 1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0, \infty)$ of positive Lebesgue measure. That is, if $A \subset [0, \infty)$ is a compact set of positive Lebesgue measure, then $M(\Lambda)$ is dense in $C[A]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

If A contains an interval, then this follows from the already-mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case when, for example, $A \subset [0, 1]$ is a Cantor type set of positive measure.

In the case that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, analyticity properties of the functions belonging to the uniform closure of $M(\Lambda)$ on A are also established.

Speculations about the above extension of Müntz's Theorem are probably as old as Müntz's Theorem itself.

Somorjai [23] and Bak and Newman [2, 19] proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0, 1]$. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0, 1]$ no matter what the underlying sequence Λ. In light of this result, in 1978 Newman [19, p. 50] raised "the very sane, if very prosaic question": Are the functions

$$\prod_{j=1}^{k}\left(\sum_{i=0}^{n_j} a_{i,j}x^{i^2}\right), \qquad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N},$$

dense in $C[0, 1]$ for some fixed $k \geq 2$? In other words, does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \ldots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied. Section 4 of this paper deals with products of Müntz spaces and answers the above question of Newman in the negative. For

(1.2)
$$\Lambda_j := \{\lambda_{i,j}\}_{i=0}^{\infty}, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \ldots, k,$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k) := \left\{p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j)\right\}.$$

A bounded Remez type inequality is established for $M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$ whenever

(1.3)
$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty, \qquad j = 1, 2, \ldots, k.$$

This obviously implies that if (1.2) and (1.3) hold and $A \subset [0, \infty)$ is a compact set of positive Lebesgue measure, then $M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$ is not dense in $C[A]$. In particular, H_4 is not dense in $C[0, 1]$, which answers Newman's problem negatively. In addition, assuming (1.2) and (1.3), our methods give an "almost characterization" of the uniform closure of $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ on A in terms of analyticity properties.

2. Bounded Remez type inequality for $M(\Lambda)$

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. For a fixed $s \in (0, 1)$, let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [0, 1] : |p(x)| \le 1\}) \ge s\},$$

where $m(\cdot)$ denotes linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[0, 1]$ given that its modulus is bounded by 1 on a subset of $[0, 1]$ of Lebesgue measure at least s. That is, how large can $\|p\|_{[0,1]}$ (the uniform norm of p on $[0, 1]$) be if $p \in \mathcal{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm\cos(n \arccos h(x))$, where h is a linear function which scales $[0, s]$ or $[1 - s, 1]$ onto $[-1, 1]$. For various proofs, extensions, and applications see [13, 14, 15, 20, 21].

We announce the following bounded Remez type inequality for $M(\Lambda)$ whose proof, which is quite difficult, will appear elsewhere.

Theorem 2.1. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $s > 0$. Then there exists a constant c depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and s (and not on ϱ, A, or the "length" of p) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

In the above theorem and throughout the paper, $\|p\|_A := \sup_{x\in A} |p(x)|$.

One might note that the existence of such a bounded Remez type inequality for a Müntz space $M(\Lambda)$ is equivalent to the non-denseness of $M(\Lambda)$ in $C[0, 1]$. We believe that this result should be a basic tool for dealing with problems about Müntz spaces. In this paper we demonstrate the power of Theorem 2.1 by settling two long-standing conjectures as fairly straightforward corrolaries.

3. Müntz's Theorem on compact sets of positive measure

Theorem 3.1. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and $A \subset [0, \infty)$ is a set of positive Lebesgue measure. Then $M(\Lambda)$ is not dense in $C[A]$. Moreover, if the gap condition*

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0 \tag{3.1}$$

holds, then every function $f \in C[A]$ from the uniform closure of $M(\Lambda)$ on A is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in A \cap [0, r_A),$$

where $r_A := \sup\{x \in [0, \infty) : m(A \cap (x, \infty)) > 0\}$ is the essential supremum of A. If the gap condition (3.1) *does not hold, then every function $f \in C[A]$ from* *the uniform closure of* $M(\Lambda)$ *on* A *can still be extended analytically throughout the region* $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$.

Proof. Suppose $f \in C[A]$, and suppose there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ which converges to f uniformly on A. Then the sequence $\{p_i\}_{i=1}^{\infty}$ is uniformly Cauchy on A. Therefore, Theorem 2.1 and the definition of r_A yield that $\{p_i\}_{i=1}^{\infty}$ is uniformly Cauchy on every closed subinterval of $[0, r_A)$. If the gap condition (3.1) holds, then the characterization of the uniform closure of $M(\Lambda)$ on A follows from the results of Clarkson and Erdős [12]. If the gap condition (3.1) does not hold, then results of Schwartz [22] yield the theorem. □

Theorem 3.2. *Suppose* $A \subset [0, \infty)$ *is a compact set of positive Lebesgue measure. Then* $M(\Lambda)$ *is dense in* $C[A]$ *if and only if* $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

Proof. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Let $f \in C[A]$. By Tietze's Extension Theorem there exists an $\tilde{f} \in C[0, 1]$ so that $\tilde{f}(x) = f(x)$ for every $x \in A$. By Müntz's Theorem there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M(\Lambda)$ which converges to $\tilde{f}$ uniformly on $[0, 1]$, hence on A. This finishes the trivial part of the theorem.

Suppose now that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Then Theorem 3.1 yields that $M(\Lambda)$ is not dense in $C[A]$. □

4. Products of Müntz spaces

We prove the following Remez type inequality for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Theorem 4.1. *Suppose* (1.2) *and* (1.3) *hold. Let* $s > 0$. *Then there exists a constant* c *depending only on* $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, *and* k (*and not on* ϱ *or* A) *so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda_2, \Lambda_2, \dots, \Lambda_k)$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

Proof. Theorem 2.1 implies that there exist constants $\alpha_j > 0$ depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, and k so that

$$m(\{x \in [y, 1] : |p(x)| > \alpha_j^{-1}|p(y)|\}) \ge 1 - y - \frac{s}{2k}$$

for every $p \in M(\Lambda_j)$ and $y \in [0, 1-s]$. Now let $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, that is, $p = \prod_{j=1}^{k} p_j$ with $p_j \in M(\Lambda_j)$. Then, for every $y \in [0, 1-s]$,

$$\begin{aligned} &m(\{x \in [y, 1] : |p(x)| > (\alpha_1\alpha_2\cdots\alpha_k)^{-1}|p(y)|\}) \\ &\quad \ge m\left(\cap_{j=1}^{k}\left\{x \in [y, 1] : |p_j(x)| > \alpha_j^{-1}|p_j(y)|\right\}\right) \\ &\quad \ge 1 - y - k\frac{s}{2k} = 1 - y - \frac{s}{2}. \end{aligned}$$

Hence $y \in [0, \inf A]$, $A \subset [0, 1]$, and $m(A) \ge s$ imply

$$m(\{x \in A : |p(x)| > (\alpha_1\alpha_2\cdots\alpha_k)^{-1}|p(y)|\}) \ge \frac{s}{2} > 0,$$

and the theorem follows with $c = \alpha_1\alpha_2\cdots\alpha_k$. □

Theorem 4.1 immediately solves Newman's problem [19].

Corollary 4.2. *Suppose* (1.2) *and* (1.3) *hold and* $A \subset [0, 1]$ *is a set of positive Lebesgue measure. Then* $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ *is not dense in* $C[A]$.

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