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Representations of solvable groups, by Olaf Manz and Thomas R. Wolf. Cambridge University Press, New York, 1993, x+302 pp., $39.95. ISBN 0-521-39739-1

One can imagine a variety of books that might be written about the representation theory of finite solvable groups. This monograph is neither a comprehensive survey nor a collection of new results. Instead the authors provide a very readable and coherent account of some of the major theorems and techniques of the subject, including complete proofs and background material. Naturally the authors emphasize results related to their own research. They also focus on results proved in the last fifteen years; expositions of some of the older results have appeared in the books of Huppert-Blackburn [2] and Isaacs [5] and in the survey articles of Berger [1] and Isaacs [6,7]. There will be something in Manz and Wolf's book for the expert, but it is aimed primarily at mathematicians and advanced students who wish to acquire a substantial knowledge of the subject as painlessly as possible.

The nongroup theorist may ask what is meant by the phrase "representation theory of solvable groups". It refers to both ordinary (characteristic zero) and modular (characteristic p) representations. It refers to both characters and modules. Unlike the representation theory of, say, the symmetric groups, it is not concerned with the computation of character tables. One is sometimes interested in describing all the irreducible characters or representations of specific solvable groups, but only as examples or counterexamples to theorems about large classes of solvable groups. Many of the techniques used in the representation theory of solvable groups extend to the larger class of p-solvable groups; for a prime p, we say a group is p-solvable if each of its composition factors is either a p-group or a p'-group.

Solvable groups (i.e., finite solvable groups) can be defined as those finite groups whose composition factors all have prime order. If G is a solvable group, one therefore has a series $1 < M_1 < \cdots < M_k = G$, where M_i is normal in M_{i+1} for $1 \leq i \leq k-1$, and M_{i+1}/M_i has prime order. Such series are of little use in the study of solvable groups. Much more useful are the chief series of G. These are, by definition, unrefinable series of the form $1 < N_1 < \cdots < N_k = G$, where each N_i is now normal in the whole group G. Each chief factor N_{i+1}/N_i is an elementary abelian p_i-group for some prime p_i. We view N_{i+1}/N_i as a vector space V_i over the field $GF(p_i)$ of p_i elements. Then G acts on V_i by conjugation, making V_i an irreducible (but not faithful) $GF(p_i)[G]$-module. If χ is an irreducible complex character of G/N_i, then the restriction of χ to V_i contains a degree-one character λ of V_i; we may view λ as an element of the dual module $V_i^* = \mathrm{Hom}(V_i, GF(p_i))$. By Clifford's Theorem, χ is induced from an irreducible character of the centralizer in G/N_i of λ.

Thus the "internal modules" V_i provide a link between the normal subgroup structure of G, the representations of G in prime characteristic, and the complex irreducible characters of G. The internal modules are of central importance in all aspects of the representation theory of solvable groups.

The problems in this area have two main sources: the structure theory of solvable groups and the (modular) representation theory of general finite groups. A long-standing problem from the first source is to show that if a group A acts on a solvable group G such that $C_G(A) = 1$ and $(|A|, |G|) = 1$, then the Fitting length of G (i.e., the minimal length of series $1 < N_1 < \cdots < N_r = G$ where each N_i is normal in G and N_{i+1}/N_i is nilpotent) is bounded above by the number of prime factors of $|A|$, counting multiplicities. When A has prime order, G must be nilpotent by a famous result of J. G. Thompson, even if G is not at first assumed to be solvable. This problem does not obviously involve representation theory, but progress on it has required a detailed analysis of the action of A and its subgroups on certain internal modules in G; see for example [8]. Another long-standing problem asks whether the number of distinct irreducible character degrees of a solvable group G is equal to or greater than the number of nonidentity terms in the series $G > [G, G] > [[G, G], [G, G]] > \cdots$. Manz and Wolf's book contains the statements and proofs of the main results on this problem.

The representation theory of general finite groups G in prime characteristic p is quite a difficult field. The irreducible complex characters of G fall into equivalence classes called p-blocks, which may be roughly defined as follows. We say that two irreducible complex characters χ and ψ of G are linked if the "reductions mod p of χ and ψ" (this can be made precise) have a common irreducible constituent. Two complex irreducible characters χ and χ' belong to the same p-block if there exists a chain $\chi = \chi_1, \chi_2, \ldots, \chi_r = \chi'$ such that χ_i and χ_{i+1} are linked for $1 \leq i \leq r-1$. There are several important conjectures about p-blocks whose proofs seem remote, even if one is willing to use the classification of finite simple groups. For p-solvable groups, however, the theory of representations in characteristic p becomes much simpler, so it is of interest to prove these conjectures for p-solvable groups. This monograph contains proofs and background material for two of these conjectures, R. Brauer's height zero conjecture and the McKay-Alperin conjecture.

With every p-block B is associated its defect group D, a certain p-subgroup of G which is defined only up to conjugacy. Brauer's height zero conjecture is the statement that D is abelian if and only if the degrees of all complex irreducible characters in B have the same p-part. A substantial portion of this monograph is devoted to the proof by Wolf and the reviewer of this conjecture for solvable groups; the extension to p-solvable groups [3] is not included.

For a group G with p-Sylow subgroup P, McKay and Isaacs conjectured that the number of complex irreducible characters of p'-degree of G equals the number of such characters of $N_G(P)$. The authors present the surprisingly short proof by Okuyama and Wajima of this conjecture for p-solvable groups; an earlier proof by Dade was much longer but gave more information. The authors include a proof, for p-solvable groups, of Alperin's block-by-block version of the conjecture.

The representation theory of solvable groups involves both elaborate theories and down-to-earth techniques. The more sophisticated theories include the character correspondences of Glauberman, Dade, and Isaacs and Gajendragadkar and Isaacs' theory of π-special characters. The authors treat some of these topics, but their main emphasis is on the down-to-earth techniques; one should add that many of the major problems in this area do not seem amenable to the more sophisticated methods.

The down-to-earth approach centers around the analysis of primitive and imprimitive modules. We say an irreducible G-module V is imprimitive if V can be written as a direct sum of subspaces which are (transitively) permuted by the action of G. If V is not imprimitive, we say V is primitive. When G is solvable, the existence of a faithful primitive G-module severely restricts the structure of G. On the other hand, if V is imprimitive, one can exploit the permutation action of G on the set of permuted direct summands of V. The proofs of many results ultimately reduce to the following "small centralizer problem"; given a G-module V, we must demonstrate the existence of a vector $v \in V$ whose centralizer in G is small in an appropriate sense. Here G ranges over some fairly general class of solvable or p-solvable groups. The module V in the small centralizer problem is usually an internal module of the original group. There are many possible meanings of "small" in the small centralizer problem. For example, we may ask if there exists $v \in V$ such that $C_G(v)$ is trivial; this is the important regular orbit problem, which plays a central role, for example, in [8]. In the proof of Brauer's height zero conjecture, we can reduce to the case of the small centralizer problem where G has a normal p'-subgroup of index p and we must show that for some $v \in V$, $C_G(v)$ is small in the sense that its order is not divisible by p.

Suppose, in the situation of the small centralizer problem, that V is imprimitive, with direct summands $V_1, \dots, V_t$ transitively permuted by the action of G. For $v = (v_1, \dots, v_t) \in V$, let $\Delta(v) = \{i \leq t : v_i = 0\}$. Then each $g \in C_G(v)$ stabilizes $\Delta(v)$, so one is led to the study of set-stabilizers in permutation groups. The authors give several results on set-stabilizers; for example, if a group G of odd order acts faithfully on a set Ω, there exists $\Delta \subset \Omega$ whose stabilizer in G is trivial. An important extreme situation in the small centralizer problem occurs when G acts transitively on the nonzero vectors of V or, in the imprimitive situation, the stabilizer in G of V_1 acts transitively on the nonzero vectors of V_1. In one of the earliest results in the representation theory of solvable groups, Huppert [4] classified the transitive solvable linear groups. Here the authors give a new proof of Huppert's result.

Many of the problems treated in Manz and Wolf's book have been completely solved, while others have reached the point where new ideas will be needed for further progress. In light of the classification of finite simple groups, one likely direction for future research is the adaptation of some of the methods here to the representation theory of nonsolvable groups.

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A. *Normal families of meromorphic functions*, by Chi-tai Chuang. World Scientific, Singapore, 1993, xi+473 pp., $68.00. ISBN 981-02-1257-7

B. *Normal families*, by Joel L. Schiff. Springer, New York, 1993, ix+236 pp., $39.00. ISBN 0-387-97967-0

Compactness of families of functions is the basis of a most important and deep method of constructing functions. For example, the simplest way to solve (locally) the differential equation

$$y' = f(x, y),$$

with f merely continuous, is to take a subsequential limit of an appropriate sequence of continuous functions.

In 1906 Paul Montel showed that a family of analytic functions on a domain $\Omega \subset \mathbb{C}$ is equicontinuous if it is uniformly bounded on compacta of Ω. This suggested that the theory of compact families would have more coherence and integrity in the context of analytic functions, and developments since Montel confirm this. For example, it is now the basis of the most standard proofs of Riemann's mapping theorem.

Montel defined a normal family as a set of analytic or meromorphic functions $\mathcal{F}$ on an open set Ω such that any sequence $f_n \subset \mathcal{F}$ contains a subsequence g_n converging uniformly on compact subsets of Ω to an analytic (meromorphic) function, possibly constant (including the infinite constant). The Montel theory (and more, as we shall see) is summarized in his monograph [9]. Sometimes one may demand that the limit function itself be in $\mathcal{F}$; in this case $\mathcal{F}$ is called *closed*, and this possibility is considered in (A).

About a decade after Montel's first findings, Fatou and Julia based their theory of iteration on his theory. It allowed sophisticated techniques to be transferred to an area which in the past had largely been devoted to ad hoc analyses of functional equations. A fine account of this is given in [1]. In view of the recent rebirth of activity in complex iteration, it is appropriate that modern treatments of normal family theory also appear. The two books under review do this, at least in the setting of one complex variable. Both presume only a standard first course, and (B) is especially conveniently arranged. They have quite different emphases but have the Montel and some post-Montel work in common.