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Cohen-Macaulay rings, by Winfried Bruns and Jürgen Herzog. Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993, xi + 403 pp., $79.95. ISBN 0-521-41068-1

The theory of Cohen-Macaulay rings and modules has become a fundamental tool over the last thirty years, both in studying commutative Noetherian rings and in algebraic geometry. It began with ideas both implicit and explicit in the extraordinary work of I. S. Cohen, who died at the age of thirty-seven, and of F. S. Macaulay, an English secondary school teacher whose accurate perception of what would prove important is astonishing (it is remarkable that [M1] appeared in 1916). The growth of this theory has taken beautiful and unexpected paths and continues with great vigor.

The book of Bruns and Herzog is a very welcome addition to the literature. It is carefully and clearly written with many examples and insightful comments, and exercises are provided. There is also some historical commentary. Much of the theory presented has not appeared in book form previously. While keeping prerequisites to a minimum, the authors have distilled results from a large, scattered literature and presented them in a most pleasant and palatable form. Besides the basics the book treats invariant theory and applications to combinatorics, determinantal rings and Hodge algebras, canonical modules and local cohomology, as well as the fundamental ideas surrounding the technique of reduction to positive characteristic and the construction of big Cohen-Macaulay modules, making the book particularly valuable to graduate students. There are sixteen pages of references.

In what follows I want to stress the geometric usefulness of the Cohen-Macaulay property, especially in intersection theory, as well as to give the flavor of some of the topics in this area and a very brief discussion of recent developments.

Algebraic geometry and intersection multiplicities. Fix an algebraically closed field K: the reader is welcome to think of K as the complex numbers $\mathbb{C}$ throughout this discussion. By an *algebraic set* in K^n we mean the common zeroes $V(f_1, \dots, f_m)$ of finitely many polynomials $f_1, \dots, f_m$ in n variables. (If there were infinitely many f's, a finite subset would generate the same ideal and have the same set of common zeroes.) When K^n itself is thought of as an algebraic set, it is denoted by $\mathbb{A}^n_K$ or, simply, $\mathbb{A}^n$.

The algebraic sets Y that are contained in an algebraic set X form the closed sets of a T_1 (but almost never Hausdorff) topology, the *Zariski* topology on X. In the case where $K = \mathbb{C}$, $X \subseteq \mathbb{A}^n$ also inherits a finer topology from the usual metric space topology on $\mathbb{C}^n \approx \mathbb{R}^{2n}$: to distinguish this topology from the Zariski topology, we shall refer to it as the *Euclidean* topology.

An algebraic set is called *irreducible*, or a *variety*, if it is not the union of two algebraic sets which are proper subsets. If $Y \subseteq X$ are varieties, we call Y a *subvariety* of X. It turns out that a nonempty algebraic set is uniquely an irredundant union of varieties, called the *irreducible components*; e.g., $V(x_1x_2) = V(x_1) \cup V(x_2)$ in the plane. By a morphism (over K) of algebraic sets $X \subseteq K^n$ to $Y \subseteq K^m$ we mean a function $f: X \to Y$ that can be described in coordinates by a formula of the form $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$, where the f_i are polynomial functions of $x_1, \dots, x_n$.

One can assign to a given algebraic set X a ring (this always means a commutative, associative ring with identity here) $K[X]$ called *the coordinate ring* of X: this ring is simply the set of all morphisms from X to $K = \mathbb{A}^1_K$ under the obvious pointwise addition and multiplication and is easily identified as the ring obtained from the polynomial functions on K^n by restricting them to X. If $\mathcal{I}(X)$ denotes the ideal of all polynomial functions that vanish on X, then $K[X] \cong K[x_1, \dots, x_n]/\mathcal{I}(X)$. There is a functorial antiequivalence (directions of arrows are reversed) between algebraic sets and their morphisms over K and finitely generated K-algebras that are *reduced* (i.e., all nilpotents are 0) and K-algebra homomorphisms. The points of an algebraic set X correspond bijectively with the maximal ideals of $K[X]$: the point P corresponds to the maximal ideal of functions that vanish at P. In fact, there is a bijective order-reversing correspondence between the ideals of $K[X]$ that are *radical* (i.e., $f^h \in I$ for some h implies that $f \in I$) and the algebraic sets Y that are subsets of X: the ideal corresponding to Y consists of all the functions in $K[X]$ that vanish on Y. In this setup, X is a variety if and only if $K[X]$ is an integral domain (cf. [F1, Ch. 1, 2]).

One can recursively give a definition of dimension for varieties as follows: points have dimension zero, and a variety has dimension n if and only if the largest dimension of any variety properly contained in it is $n-1$. With these conventions, one has $\dim K^n = n$. When $K = \mathbb{C}$, $\dim X$ is half its (real) topological dimension: in fact, if $\dim X = d$, then X is the union of a (closed) subvariety of dimension smaller than d and a set which, in the Euclidean topology, is a $2d$-manifold (cf. [Mum, §1A, B]). One can extend this notion of dimension to all algebraic sets by letting the dimension be, in general, the supremum of the dimensions of the irreducible components. This leads one to define the (Krull) *dimension* $\dim R$ of a ring R as the supremum of lengths h of chains $P_0 \subset \cdots \subset P_h$ of prime ideals in R. (When we refer to vector space dimension instead, we shall always use the field as a subscript.)

Evidently, there is a tautological sense in which every property of algebraic sets defines a corresponding property for finitely generated reduced K-algebras, and conversely. The very natural geometric notion of dimension for algebraic sets leads to a notion that is actually well defined for all commutative rings with identity, while the property of being a variety corresponds to the ring-theoretic property of being a domain. What we want to do next is discuss intersection multiplicities with the goal of giving geometric motivation for the Cohen-Macaulay property.

Given two varieties X, Y in $\mathbb{A}^n$ having an isolated point of intersection P and such that $\dim X + \dim Y = n$ (in this situation $\dim X + \dim Y \leq n$, or else X and Y are forced to have a bigger intersection), it is a classical problem to decide what intersection multiplicity should be assigned to P.

There is a very simple solution to the problem of defining intersection multiplicities if $X = V(f)$ and $Y = V(g)$ are curves in the plane. First, if P

is a point of the variety X, let $\mathcal{O}_{X,P}$ denote the *local ring* of X at P: it is obtained by adjoining to $K[X]$ inverses for all elements of $K[X]$ not vanishing at P. Then simply let the intersection multiplicity be $\dim_K \mathcal{O}_{\mathbb{A}^2,P}/(f, g)$, where (f, g) denotes the ideal generated by f and g. Notice that we have used the local ring at P in the numerator rather than $K[x_1, x_2]$ itself. The purpose of the localization is to get rid of any contribution from points of intersection, other than P, of the two curves. It suffices to invert enough elements of $K[x_1, x_2]$ so that for every point of $V(f, g) - \{P\}$, at least one of the inverted elements vanishes at that point. In particular, if $V(f, g) = \{P\}$, one may use $K[x_1, x_2]$ itself in the numerator. (An alternative, quite generally, is to use the formal power series ring $K[[x_1, x_2]]$.) When the two curves have no common tangent, this number can be interpreted as the product of the numbers of tangent lines, at P, to the two curves (counted with suitable multiplicities!). When the two curves have a common tangent, the intersection multiplicity is larger and reflects the order of tangency. For example, the intersection multiplicity of $V(x_2)$ and $V(x_2 - x_1^d)$ at the origin in $\mathbb{A}^2$ is $\dim_K K[x_1, x_2]/(x_2, x_2 - x_1^d) = \dim_K K[x_1]/(x_1^d) = d$.

One important indication that this is a good notion of intersection multiplicity is that the following statement, known as Bezout's theorem, is true when points of intersection are counted with multiplicities this way: If $f(x_1, x_2)$ and $g(x_1, x_2)$ are polynomials of respective degrees d, $e \geq 1$ such that $V(f)$ and $V(g)$ have no common component (i.e., f, g are relatively prime) and such that the leading forms of f and g are relatively prime (this condition means that the curves have no common asymptote, which would correspond to a point of intersection "at infinity" if we pass to the projective plane), then the number of points of intersection of $V(f)$ and $V(g)$, counting multiplicities, is de. (If one works in the projective plane, the statement becomes simpler (cf. [F1, Ch. 5, §3]).

The definition of intersection multiplicity for curves in the plane that we have given turns out to be very easy to use in calculations (cf. [F1, Ch. 3]). However, it does not appear to be well motivated geometrically, and, in higher dimension it is, in general, the wrong notion. To describe the correct notion, we first study the intersection of varieties X, $Y \subseteq \mathbb{A}^n_{\mathbb{C}}$ at the origin P (we can always make a change of coordinates so that a given point of intersection is the origin), in the special case where Y is defined by linear forms $L_1, \ldots, L_d$, so that $Y = V(L_1, \ldots, L_d)$. As usual we are assuming that $\dim X + \dim Y = n$ and that P is an isolated point of the intersection. We have that $\dim X = n - (n - d) = d$. The idea is to count the points of intersection after we move the linear space Y slightly. Let $\epsilon_1, \ldots, \epsilon_d$ be very small but nonzero complex numbers. It turns out that for all sufficiently small choices of $(\epsilon_1, \ldots, \epsilon_d)$ lying off a proper closed subvariety of $\mathbb{A}^d_{\mathbb{C}}$, the cardinality of the set of points of the intersection $X \cap V(L_1 - \epsilon_1, \ldots, , L_d - \epsilon_d)$ that are close to P in the Euclidean topology is a constant, and this is the number we want to use as the intersection multiplicity. (Try this for $V(x_2) \cap V(x_2 - x_1^d)$.)

This notion is very natural and geometric. Moreover, the same number can be described algebraically as follows: let z_i be the image of L_i in $R = \mathcal{O}_{X,P}$, and let Q be the ideal of R generated by the z_i. One can show that $H_{R,Q}(t) = \dim_{\mathbb{C}}(R/Q^t)$ is a polynomial in t of degree d, whose leading coefficient is

$\mu/d!$, where μ is the desired intersection multiplicity. This gives a way of defining the intersection multiplicity when the field is not necessarily $\mathbb{C}$. Of course, we have not proved that this is the same number when $K = \mathbb{C}$.

In the general case, when Y is not necessarily a linear space, one can use the trick, helpful in many kinds of geometry, of studying instead the intersection in $\mathbb{A}^{2n}$ of $X \times Y$ with diagonal $\Delta = \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$, where $\Delta = \{(\lambda, \lambda) : \lambda \in \mathbb{A}^n\}$. There is a set-theoretic bijection $X \cap Y \to (X \times Y) \cap \Delta$ given by $\lambda \mapsto (\lambda, \lambda)$, and it turns out that thinking this way gives a good notion of intersection multiplicity as well. Since Δ is defined by linear forms, we are now in the case that we have already handled.

This is a bit complicated. It would be pleasant if one could get at the intersection multiplicity of X and Y at a point P in $\mathbb{A}^n$ using the analogue of the formula that we have for plane curves, i.e., $\dim_K \mathscr{O}_{\mathbb{A}^n, P}/(\mathscr{I}(X) + \mathscr{I}(Y))$. To see that the two notions are different in general, consider the following:

Example 1. Let X denote the subvariety of $\mathbb{A}^4_{\mathbb{C}}$ consisting of all points whose coordinates have the form (s^2, t, s^3, st) for some choice of s and t in $\mathbb{C}$. It is not difficult to show that $X = V(x_1^3 - x_3^2, x_1x_4 - x_2x_3, x_3x_4 - x_1^2x_2, x_1x_2^2 - x_4^2)$ (hence, X is an algebraic set) and that the four polynomials given generate $\mathscr{I}(X)$. Let $Y = V(x_1, x_2)$. Then X and Y meet only at the origin, P, and the quotient $K[x_1, x_2, x_3, x_4]/(\mathscr{I}(X) + \mathscr{I}(Y)) \cong \mathbb{C}[x_3, x_4]/(x_3^2, x_3x_4, x_4^2)$, which has dimension 3 over $\mathbb{C}$. However, if ϵ_1 and ϵ_2 are nonzero, the number of points of intersection of X with $V(x_1 - \epsilon_1, x_2 - \epsilon_2)$ is easily seen to be only two, and the intersection multiplicity is two.

We are now led to the definition of a Cohen-Macaulay ring. By a "local ring" (R, m, L) we mean a Noetherian ring R with a unique maximal ideal m: the notation means that $L = R/m$. Sometimes we omit specifying L (or m and L). It turns out that if a local ring has Krull dimension d, then there are d elements $x_1, \dots, x_d$ in m such that the ring $R/(x_1, \dots, x_d)$ has Krull dimension 0: this is equivalent to the condition that $m^t \subseteq (x_1, \dots, x_d)$ for some t, and no fewer than d elements can be used. If $d = \dim R$ and $\dim R/(x_1, \dots, x_d) = 0$, we say that $x_1, \dots, x_d$ is a *system of parameters* for R. In this case, we would like to refer to something like the dimension of the L-vector space $R/(x_1, \dots, x_d)$, but this does not make sense because the ring may not contain a copy of its residue class field L. However, we can get around this by using the notion of "length": the length of a finitely generated R-module M such that $m^tM = 0$ is defined as $\sum_{i=0}^{t-1} \dim_L(m^iM/m^{i+1}M)$ and is denoted $\ell(M)$. We may then define the *multiplicity* of the system of parameters $x_1, \dots, x_d$ to be the integer μ such that the leading coefficient of the degree d polynomial in t that agrees with $\ell(R/(x_1, \dots, x_d)^t)$ for all $t \gg 0$ is $\mu/d!$.

Let M be an R-module. A sequence of elements $x_1, \dots, x_d \in R$ is called a *regular sequence* on M if $(x_1, \dots, x_d)M \neq M$ and for $0 \leq i \leq d-1$ the image of x_{i+1} is not a zerodivisor in $M/(x_1, \dots, x_i)M$. The main case here is when $M = R$, in which case the first requirement is simply that the elements generate a proper ideal of R.

If I is an ideal of a ring R, we may form an *associated graded ring*, $\mathrm{gr}_I R = \oplus_{t=0}^{\infty} I^t/I^{t+1}$, whose t^{th} graded piece, $t \geq 0$, is I^t/I^{t+1}, where if $u = [r]$ with $r \in I^s$ has degree s and $u' = [r']$ with $r' \in I^t$ has degree t, then $uu' = [rr']$ in

I^{s+t}/I^{s+t+1}. If $x_1, \ldots, x_d$ generate I, then $[x_1], \ldots, [x_d]$ in I/I^2 generate $\mathrm{gr}_I R$ as an (R/I)-algebra.

Proposition-Definition 2. *Let (R, m, L) be a local ring of Krull dimension d. The following conditions are equivalent:*

(a) *Some (equivalently, every) system of parameters is a regular sequence on R.*
(b) *For some (equivalently, every) system of parameters $x_1, \ldots, x_d$, the multiplicity of the system of parameters $x_1, \ldots, x_d$ is equal to $\ell(R/(x_1, \ldots, x_d))$.*
(c) *For some (equivalently, every) system of parameters $x_1, \ldots, x_d$, if Q denotes $(x_1, \ldots, x_d)R$, then $\mathrm{gr}_Q R$ is a polynomial ring in the variables $[x_i]$ over R/Q.*

If R satisfies these equivalent conditions, it is called a Cohen-Macaulay ring.

See [S, Th. 3, p. IV-21]. The precise connection of this property with intersection multiplicities is given below:

Theorem 3. *Let $X \subseteq \mathbb{A}_K^n$ be a variety with $\dim X = d$, let $P \in X$, and let $R = \mathcal{O}_{X,P}$. Then the following conditions are equivalent:*

(a) *R is Cohen-Macaulay.*
(b) *For one linear space Y of dimension $n-d$ such that P is an isolated point of intersection of X and Y (such spaces Y always exist), the intersection multiplicity of X and Y at P is $\dim_K \mathcal{O}_{\mathbb{A}^n,P}/(\mathcal{I}(X) + \mathcal{I}(Y))$.*
(c) *For every subvariety of Y of dimension $n-d$ such that $\mathcal{O}_{Y,P}$ is Cohen-Macaulay and P is an isolated point of intersection of X and Y, the intersection multiplicity of X and Y at P is $\dim_K \mathcal{O}_{\mathbb{A}^n,P}/(\mathcal{I}(X)+\mathcal{I}(Y))$.* (Cf. [S, Cor., p. V-20].)

The restriction to the case of varieties is not really needed but is made here to avoid technicalities.

In this intersection-theoretic context, the Cohen-Macaulay property appears very naturally — indeed, it is thrust upon us. The fact that the naïve formula $\dim_K \mathcal{O}_{\mathbb{A}^n,P}/(\mathcal{I}(X) + \mathcal{I}(Y))$ for intersection multiplicities is always valid for $n = 2$ when X and Y are curves may be viewed as a consequence of the easily verified fact that all one-dimensional reduced local rings are Cohen-Macaulay.

The ring $\mathbb{C}[X]$, where X is the variety of Example 1, is not Cohen-Macaulay when localized at the origin. In fact, $\mathbb{C}[X] \cong \mathbb{C}[s^2, t, s^3, st] \subseteq \mathbb{C}[s, t]$. In the localization, s^2 and t form a system of parameters, but t is a zero-divisor modulo s^2, because $s^3(t) = st(s^2)$, but st is not a multiple of t in this ring, since s is missing. (We are using characterization (a) from Proposition 2.)

Comparison with regular rings. The process of localization is very general. If R is a commutative ring, we may define the localization R_Q of R at any prime ideal Q thus: if W is the complement of Q, $R_Q = R[x_r : r \in W]/(rx_r - 1)$, where the x_r are indeterminates indexed by W. This R-algebra has been provided with an inverse for every element of W in a universal way. We can then define a Noetherian ring to be *Cohen-Macaulay* if all of its localizations at prime ideals (equivalently, maximal ideals) are Cohen-Macaulay local rings.

Near a point P of a d-dimensional variety X in $\mathbb{A}_{\mathbb{C}}^n$, it may be possible to use the analytic implicit function theorem to "solve" for $n-d$ of the variables in terms of the remaining d in the equations defining X. Then P is called a *regular*, *smooth*, *nonsingular*, or *simple* point of X. Near P, X has the structure of a d-dimensional analytic manifold. In fact, the regular points of X form a Zariski open set that has the structure of an analytic manifold. The d-dimensional local ring (R, m, K) of a variety at a point P is the local ring at a regular point if and only if it satisfies the following three equivalent properties:

(a) The maximal ideal m of R is generated by d elements (iff $\dim_K m/m^2 = d$).
(b) The maximal ideal m of R is generated by a regular sequence.
(c) $\mathrm{gr}_m R$ is isomorphic with a polynomial ring in d variables over K.

A Noetherian ring is called *regular* if its local rings at maximal ideals (equivalently, at all prime ideals) satisfy these equivalent conditions. A polynomial ring or formal power series ring in finitely many variables over a field is regular. It is worth noting that every Noetherian local ring (R, m, K) has a completion in its m-adic topology (define the distance between r and s when $r \neq s$ to be 2^{-h}, where $r - s \in m^h - m^{h+1}$), which is a (Noetherian) local ring. Completion does not affect the property of being Cohen-Macaulay, nor does it affect the property of being regular. The only complete regular local rings containing a field are isomorphic to formal power series rings.

It turns out that in many cases Noetherian rings can be viewed as finitely generated modules over regular subrings. We note the following results:

Theorem 4 (a) (Noether normalization). *Let R be a finitely generated algebra of dimension d over a field K. Then there are d algebraically independent elements $x_1, \ldots, x_d$ in R such that the ring R is finitely generated as a module over its subring $A = K[x_1, \ldots, x_d]$. If R is graded by the nonnegative integers with $R_0 = K$, then these elements may be chosen to be homogeneous of the same degree; if the field is infinite and R is generated by forms of degree one, they may be chosen to have degree one.*

(b) (I. S. Cohen). *Let R be a complete local ring containing a field. If $x_1, \ldots, x_d$ is a system of parameters, then R is finitely generated as a module over its subring A of formal power series over K in $x_1, \ldots, x_d$, which is isomorphic to a formal power series ring in d variables over K and so is regular* (cf. [C]).

Regular rings are in many ways the best understood. Therefore the following alternative characterizations of the Cohen-Macaulay property are very useful.

Proposition 5. *Let $R \subseteq S$ be Noetherian rings such that R is regular and S is finitely generated as an R-module.*
(a) *If R, S are local, then S is Cohen-Macaulay if and only if it is a free R-module.*
(b) *If R, S are nonnegatively graded finitely generated K-algebras with $R_0 = S_0 = K$ and the inclusion preserves degree, then S is Cohen-Macaulay if and only if it is free as an R-module.*
(c) *If S is a domain and R is finitely generated over K, then S is Cohen-Macaulay if and only if it is a projective R-module* (*i.e., a direct summand of a free R-module*).

Since finitely generated K-algebras are always finitely generated modules over polynomial subrings (and in a degree-preserving manner in the graded case), while complete local rings containing a field are likewise always finitely generated modules over a formal power series subring, these criteria are widely applicable.

We revisit Example 1. The ring $R = \mathbb{C}[s^2, t, s^3, st]$ is finitely generated over the graded polynomial subring $R_0 = \mathbb{C}[s^2, t]$, but it is not a free module: every set of generators for R over R_0 needs at least three elements, and there will be some relation, e.g., 1, s^3, st generate, i.e., $R = R_0 + R_0s^3 + R_0st$, but $t(s^3) - s^2(st) = 0$.

It is worth noting that if R is regular local and $f_1, \ldots, f_h$ form part of a system of parameters, then $R/(f_1, \ldots, f_h)$ is always Cohen-Macaulay, and is called a *local complete intersection.* If $h = 1$, such a ring is called a *local hypersurface.* These are among the most important examples of Cohen-Macaulay rings.

Invariant theory. An immense effort over the years has gone into the study of rings of invariants or fixed rings of actions of groups on rings, especially when the group G is a subgroup of the general linear group $GL(n, K)$ acting in the obvious way on a polynomial ring R in n variables. If K has characteristic 0 and G is reductive (every representation is completely reducible: these groups include semisimple groups, algebraic tori, and finite groups), one hopes that the fixed ring R^G will have good properties. It is a finitely generated K-algebra under these hypotheses. The surprising fact is that while it may be very complicated, the ring of invariants is always Cohen-Macaulay (cf. [HR, Bou, Ke, HH1, HH3, HH4]).

To get an idea of the complexity of the examples, let X, Y be $r \times n$ and $n \times s$ matrices of indeterminates over $\mathbb{C}$, and let $G = GL(n, \mathbb{C})$ act on the polynomial ring R over $\mathbb{C}$ generated by the entries of X and Y so that $\alpha \in G$ maps the entries of X and Y to the entries of $X\alpha^{-1}$ and αY, respectively. The ring of invariants is generated over $\mathbb{C}$ by the entries of the matrix XY. It turns out that $R^G \cong \mathbb{C}[Z]/I_{n+1}(Z)$, where Z is an $r \times s$ matrix of new indeterminates and $I_{n+1}(Z)$ is the ideal generated by the size $n+1$ minors of Z. This ring is not a unique factorization domain even when $r = s = 2$ and $n = 1$. It is far from obvious that it is Cohen-Macaulay, but this follows from the main result of [HR].

The book under review gives a reasonably elementary proof of the fact that rings of invariants of reductive groups acting linearly on polynomial rings in characteristic 0 are Cohen-Macaulay: the argument was developed by F. Knop based on the tight closure proofs given in [HH1, HH4]. This proof, like the original argument in [HR], uses reduction to positive characteristic. (Determinantal rings like $K[Z]/I_{n+1}(Z)$ are actually known to be Cohen-Macaulay in all characteristics: see [HoE], for example.)

Many important examples of Cohen-Macaulay rings arise as rings of invariants of reductive groups, and the fact that such rings are Cohen-Macaulay is very useful. Note, however, that it is *not* true that if R is Cohen-Macaulay and G is reductive, then R^G must be Cohen-Macaulay.

Hilbert functions and combinatorics. Throughout this section let K be a field, and let R be a *standard* graded K-algebra, by which we mean that R is finitely generated over K by forms of degree one and that $R_0 = K$. The function on the nonnegative integers $t \mapsto \dim_K R_t$, which is called the *Hilbert function* of

R, agrees with a polynomial $H_R(t)$ in t of degree $\dim R - 1$ for all $t \gg 0$, and this polynomial is called the *Hilbert polynomial.* For simplicity we assume that K is infinite.

If R is Cohen-Macaulay of dimension n, there is a particularly simple way to understand the Hilbert polynomial of R. In this case R is a *free* module over a subring R_0, where R_0 is a polynomial ring in forms of degree one, and the free basis for R over R_0 will be given by a finite number of forms of nonnegative degree. If the degrees are given by $e_1, \dots, e_h$, then, since the Hilbert polynomial of the polynomial ring is given by $\binom{t+n-1}{n-1}$, the Hilbert polynomial of the graded free module is given by $\sum_{j=1}^{h} \binom{t+n-e_j-1}{n-1}$. It is also not hard to keep track of the precise Hilbert *function.*

There is a useful way of assigning a standard graded K-algebra to a finite abstract simplicial complex Δ (cf. [Re, St1-2]). Let $x_1, \dots, x_n$ be indeterminates corresponding to the vertices of Δ, and let $K[\Delta]$ denote the ring $K[x_1, \dots, x_n]/I$, where I is generated by all monomials such that the set of variables occurring in the monomial is not a face of Δ. Let f_i be the number of faces of dimension i in Δ, $0 \le i \le d = \dim \Delta$. Then $(f_0, \dots, f_d)$ is called the *f-vector* of Δ. It is easy to show that the Hilbert function of Δ is $t \mapsto \sum_{i=0}^{d} f_i \binom{t-1}{i}$ for $t \ge 1$ by a simple counting argument. The Cohen-Macaulay property of $K[\Delta]$, when K has a specified characteristic, is a topological invariant of the geometric realization $|\Delta|$ of Δ. (The condition in [Re] for $K[\Delta]$ to be Cohen-Macaulay is that if Λ is Δ or the link of any face σ of Δ (the *link* of σ is the set of faces $\tau \in \Delta$ such that $\tau \cup \sigma \in \Delta$ and $\tau \cap \sigma = \varnothing$), then the reduced simplicial homology modules of Λ with coefficients in K must vanish, except possibly in degree $\dim \Lambda$. This condition is shown to be a topological property of $|\Delta|$ in [Mun]. It can depend on the characteristic: a triangulation of a real projective plane yields a Cohen-Macaulay ring except in characteristic 2.)

Inequalities due to Macaulay [M2] describing the behavior of the Hilbert function of a standard graded Cohen-Macaulay ring then give inequalities that the numbers of faces of a simplicial complex must satisfy when $K[\Delta]$ is Cohen-Macaulay. This idea was used by Richard Stanley [St1-2] to prove a conjecture (the Upper Bound Conjecture) concerning f-vectors occurring in triangulations of spheres; when Δ is a triangulation of a sphere, $K[\Delta]$ is Cohen-Macaulay. (See also [St3], where the necessity of a characterization of the f-vectors is proved.) Although other methods were eventually developed for attacking these problems, it was the theory of Cohen-Macaulay rings that led to the first solution.

More geometry. We give here two more connections between the Cohen-Macaulay property and algebraic geometry: these require basic familiarity with projective varieties and cohomology of coherent sheaves (cf. [Ha]). Readers without this background should skip to the next section.

First, let R be a standard graded K-algebra that is a domain integrally closed in its field of fractions. In general, if $\dim R \ge 3$, R need not be Cohen-Macaulay. Let X be the projective variety $\operatorname{Proj} R$ associated with R, and let $\mathscr{L} = \mathscr{O}_X(1)$, which is a certain line bundle on X. Then R is Cohen-Macaulay if and only if $H^i(X, \mathscr{L}^{\otimes t}) = 0$ for all integers t, $1 \le i < \dim X$. This illuminates the geometric significance of the Cohen-Macaulay property.

Second, it is for projective varieties (and, more generally, schemes) whose local rings are Cohen-Macaulay that one has an especially palatable form of Serre duality; e.g., when X is a projective variety of dimension d whose local rings are Cohen-Macaulay, there is a torsion-free sheaf ω_X on X called the *canonical* or *dualizing sheaf* such that for every locally free coherent sheaf $\mathscr{F}$ on X, $H^i(X, \mathscr{F}) \cong H^{d-i}(X, \omega_X \otimes_{\mathscr{O}_X} \mathscr{F}^\vee)^*$, where $^\vee$ indicates the dual into the structure sheaf and * indicates the vector space dual over K. This duality, which is closely related to local duality, is a generalization of Roch's part of the Riemann-Roch theorem for curves. There is a detailed treatment in [AK].

Big Cohen-Macaulay modules and local homological questions. For simplicity, let R be a complete local domain. In this concluding section, we want to discuss the fact that when R is not Cohen-Macaulay, it turns out to be of considerable interest to know whether there exists an R-module M such that some system of parameters for R is a regular sequence on M. This has important consequences even when M is not restricted to be a finitely generated module. Such "big Cohen-Macaulay modules" are known to exist if R contains a field (or if $\dim R \leq 2$). This was first proved in [Ho1]. The proof in the equal characteristic zero case uses the technique of reduction to characteristic $p > 0$. The argument is given in the work under review. One reason for the interest in this question is that the existence of these modules enables one to answer certain questions raised by Auslander, Bass, Peskine and Szpiro, and others in a simple way (cf. [Au, B, PS1-2, Ho1-2, and Ro1-2]: e.g., let R be local and M be a finitely generated nonzero R-module. One may ask (1) if M has a finite right injective resolution, then must R be Cohen-Macaulay? and (2) if M has a finite left projective resolution and $r \in R$ is a zerodivisor on R, then must r be a zerodivisor on M? The first substantial progress on these questions was made in [PS1-2]. A solution in the equicharacteristic case is given in [Ho1]. Both (1) and (2) have now been settled [Ro1-2] even for local rings not necessarily containing a field using sophisticated ideas from intersection theory (cf. [F2]), but other related questions remain open. An intriguingly simple one is this: if R is a regular Noetherian ring and S is a module-finite extension ring, is R a direct summand of S as an R-module? This is known if R contains a field or $\dim R \leq 2$, but is open even if $R = \mathbb{Z}[x, y]$. It would follow if the existence of big Cohen-Macaulay modules could be proved for all complete local domains (cf. [Ho1-2]). (It is worth mentioning that one of the old homological questions, the rigidity of Tor (see [Au, PS1]) has recently been settled negatively [He].)

Recently, it has been shown [HH2] that if R is a complete local domain of characteristic p, then R^+, the integral closure of R in an algebraic closure of its fraction field, is actually a big Cohen-Macaulay algebra for R. This enables one to show (cf. [HH3]) that if $(R_1, m_1) \to (R_2, m_2)$ is a homomorphism of local rings containing a field such that m_1 maps into m_2, then there is a commutative diagram

$$\begin{array}{ccc} B_1 & \to & B_2 \\ \uparrow & & \uparrow \\ R_1 & \to & R_2 \end{array}$$

such that for $i = 1, 2$, B_i is a big Cohen-Macaulay algebra for R_i, and this fact can be used to prove not only the homological results that follow from the existence of big Cohen-Macaulay modules but also the fact, discussed earlier, that rings of invariants of reductive groups over fields of characteristic zero acting on regular rings are Cohen-Macaulay.

The homological theorems, the invariant theory result, as well as many other theorems can be obtained, often in a vastly improved form (and with rather short proofs) using the recently developed theory of *tight closure*: see [HH1, HH4].

I hope this taste of the theory of Cohen-Macaulay rings and modules has whetted the reader's appetite. The book by Bruns and Herzog is an excellent introduction.

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MELVIN HOCHSTER
UNIVERSITY OF MICHIGAN

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Multivalued differential equations, by Klaus Deimling. de Gruyter, Berlin and New York, 1992, 260 pp., $59.00. ISBN 3-11-0132125

Although some papers on multivalued differential equations, or differential inclusions, appeared in the literature before the middle of the century, the subject began to interest mathematicians seriously in the 1960s. Several motivations concurred. On one hand there was the interest in control theory and in optimal control. In an ordinary differential equation such as $x'(t) = f(t, x(t))$, an additional parameter u (the control) is introduced, and one considers the controlled system $x' = f(t, x, u)$, $u \in U$, where U is the set of admissible controls. Hence, at a given time t and a given state $x(t)$, several velocities are possible: $x'(t)$ has to belong to the set $\{f(t, x(t), u) : u \in U\}$. So what really counts for the dynamics is this set, which can be described by a set-valued map F: from $\mathbb{R} \times \mathbb{R}^n$ to the nonempty subsets of $\mathbb{R}^n$, and the controlled differential equation becomes an inclusion, i.e.,

$$x'(t) \in F(t, x(t)).$$

The above has been an important motivation for studying differential inclusions and has led to the simplification of some proofs on the existence of optimal controls [8]. At about the same time, interest arose in nondifferentiable functions. A continuous but not differentiable convex function admits subdifferentials. Minima of a functional of the calculus of variations of the kind

$$\int_I g(t, x(t), x'(t))\, dt$$