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Measure theory and fine properties of functions by L.C. Evans and R. Gariepy. CRC Press, Boca Raton, Ann Arbor, and London, 1992, viii + 268 pp., $59.95. ISBN 0-8493-7157-0

The 1950s saw many beautiful developments concerning domains with very general boundaries and functions of bounded variation in n variables, for example, the Gauss-Green theorems of Federer [F1] and DeGiorgi [DG1], the co-area formula of Fleming-Rishel [F-R], and DeGiorgi's theory [DG2] of sets of finite and least perimeter. Analytic and geometric properties of functions of bounded variation have become important and basic to many problems in analysis, partial differential equations, and applied mathematics.

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A locally integrable function f on a domain Ω in $\mathbf{R}^n$ is in $\mathrm{BV}(\Omega)$ (i.e. of *bounded variation in* Ω) if

$$\sup\{\int_\Omega f \operatorname{div}\phi\, dx \;:\; \phi \in \mathscr{C}_0^\infty(\Omega, \mathbf{R}^n),\ \|\phi\|_{L^\infty} \le 1\} < \infty .$$

Thus, by the Riesz Representation Theorem, the distribution gradient corresponds to a vector-valued Radon measure $[Df]$ in the sense that the above integral can be evaluated as

$$-\int_\Omega \phi \cdot d[Df] .$$

In the special case that the measure $[Df]$ is absolutely continuous and summable with respect to Lebesgue measure so that $[Df] = \nabla f\, dx$ for some function $\nabla f \in L^1(\Omega, dx)$, the function f belongs to the Sobolev space $W^{1,1}(\Omega)$. However, there are many other interesting BV functions. For example, the characteristic function of a smooth bounded domain with compact closure in Ω belongs to $\mathrm{BV}(\Omega)$ but not to $W^{1,1}(\Omega)$. More generally, one says that a measurable set A has *finite perimeter* if its characteristic function χ_A belongs to BV. While the topological boundary ∂A may be quite large, even containing an interior, the gradient measure $[D\chi_A]$ is actually carried by a set $E \subset \partial A$ of finite Hausdorff $\mathscr{H}^{n-1}$ measure that is countably (n-1) rectifiable, that is, a subset of a countable union of $\mathscr{C}^1$ submanifolds and an $\mathscr{H}^{n-1}$ null set. The measure $[D\chi_A]$ is absolutely continuous with respect to the restriction of $\mathscr{H}^{n-1}$ to E, and the Radon-Nikodym derivative is $\mathscr{H}^{n-1}$ almost everywhere, a unit vector normal to E. Then, for a $\mathscr{C}^1$ vectorfield on $\mathbf{R}^n$, the Gauss-Green theorem holds true for an arbitrary subset A of finite perimeter.

For $n = 1$, a general BV function f equals a.e. a function $\tilde{f}$ of finite total variation, in the classical sense, where the total variation of $\tilde{f}$ on an interval A is the total mass $\|[Df]\|(A)$ of A with respect to the corresponding positive measure $\|[Df]\|$. The higher dimension version of this relation is the beautiful *co-area formula*

$$\|[Df]\|(A) = \int_{-\infty}^{\infty} \|[D\chi_{\{f>t\}}]\|(A)\, dt$$

giving the total variation on A as an integral of the perimeter of the sublevel sets in A. If case f is Lipschitz, this relative perimeter equals $\mathscr{H}^{n-1}(A \cap f^{-1}\{t\})$ for almost all t. The co-area formula allows one to show that the Sobolev inequality

$$\left[\int |f|^{\frac{n}{n-1}}\, dx\right]^{\frac{n-1}{n}} \le c_n \|[Df]\|(\mathbf{R}^n)$$

for $f \in \mathrm{BV}(\mathbf{R}^n)$ is equivalent to the isoperimetric inequality

$$\mathscr{H}^n(A)^{\frac{n-1}{n}} \le c_n \|[D\chi_A]\|(\mathbf{R}^n)$$

for a bounded set A of finite perimeter. For a bounded Lipschitz domain Ω, there are similarly related a Poincaré inequality for $f \in \mathrm{BV}(\Omega)$ and a relative isoperimetric inequality for subsets with finite perimeter in Ω.

BV functions also enjoy many remarkable pointwise properties. Recall that, for $n = 1$, $\tilde{f}$ has left and right limits everywhere and at most a countable number of jump discontinuities. In higher dimensions, f is approximately continuous off a countably $(n-1)$ rectifiable set E. Moreover, at $\mathscr{H}^{n-1}$ almost all points of E , f has approximate "one-sided" limits.

All these facts are proven directly and elegantly in the book of Evans and Gariepy. In addition, one finds some nice results concerning many more special classes of functions. Several pointwise properties for a general Sobolev function in $W^{1,p}$ are established. In particular, there is the Federer-Ziemer theorem that the p-Lebesgue set of a suitable representative has complement that is small, having p-capacity zero and hence Hausdorff dimension at most $n-p$. Included also are the important theorems of Rademacher on the differentiability a.e. of Lipschitz functions, of Alexandrov on the twice differentiability a.e. of convex functions, and of Whitney on the extension of $\mathscr{C}^1$ functions.

This book was developed from a series of lecture notes, and retains the lively spirit and rigor of a carefully taught course. The choice of topics is remarkably economical and the proofs direct, elegant, and complete. Every topic is important, and a random bite anywhere in the book gives all "beef". The whole setup, including the organization, references, bibliography, notes, and catalogue of notations is very user-friendly. The choice of references, like the text, is *not* excessively long (it is only one page) but is very useful. Moreover, by combining together the bibliographies of all of these references, one obtains a fairly comprehensive list. There are very few misprints, and an updated errata sheet is posted in an AMS-TeX file available via the `math.berkeley.edu` gopher server in the directory `Preprints/L_C_Evans`.

The main criticism is the lack of exercises. Because of the elegance of the presentation, the beginner may be lulled into a false sense of security concerning BV and Sobolev functions. On the one hand, it is fair that the teacher of a course on these topics using such a great text have to do *some work* himself or herself and come up with some good problems. On the other hand, it is true that the presence of the excellent collection of exercises in the widely read Rudin's *Real and Complex Analysis* has served well many generations of mathematicians.

It may also be useful to make a few very brief comments on some of the other related introductory texts on BV. The encyclopedic book of Federer [F2] gives a very comprehensive treatment of BV, Hausdorff measures, and geometric measure theory in general. This book is a standard for the field and is essentially self-contained. However it is rough going for a beginner, and there are now, including Evans-Gariepy, several introductory treatments of BV developing other important related topics. The Australia notes of Giusti [G] aimed well at exposing DeGiorgi's theory, including recent developments, on sets of finite and least perimeter. Many applications to P.D.E. are in Volpert's text [V]. Leon Simon's notes [S] cover much BV theory while studying aspects of geometric measure theory relevant to the calculus of variations, the rectifiable currents of Federer and Fleming, and varifolds of Almgren and Allard. William Ziemer's nice book [Z] was developed, I believe, essentially concurrently with the present book. It gives particularly good treatments of symmetrization and the Poincaré inequalities.

In summary, the subject matter and presentation of the Evans-Gariepy book are excellent for the first analysis course after a course in Lebesgue integration. The reviewer apologizes for being so late with this review. One advantage has been the opportunity of getting reports from colleagues who have also already used this book in their graduate courses. To quote one, "It's simply great."

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Mathematical theory of incompressible viscous fluids, by Carlo Marchioro and Mario Pulvirenti. Applied Mathematical Sciences, vol. 96, Springer-Verlag, Berlin and New York, 1994, xi+283 pp. $49.00. ISBN 0-387-94044-8

This book is about fluids which are ideal—incompressible and inviscid. Incompressibility is the property of volume preservation: as the fluid flows, any region in it conserves exactly its volume. Inviscid fluids are free of internal friction; in particular they do not wet boundaries. Are such idealizations reasonable? Are they necessary? Or are they irrelevant? This subject is close to the heart of at least three professions: engineering, mathematics, and physics. As is often the case, practical knowledge about fluids precedes in many regards theoretical knowledge. In recent years, however, boundaries between the disciplines have become more blurred: experimental and computational advances made the mathematicians' and theoretical physicists' preoccupations closer to engineering purposes. The incompressible Euler equations describe ideal fluids. "Real" Newtonian fluids are described by the Navier-Stokes equation. The Euler equations capture the main feature of the Navier-Stokes equation—its nonlinearity. Friction is represented in the Navier-Stokes equations by a linear term, but this is the term with the highest number of derivatives. Thus the Navier-Stokes equations are a singular perturbation of the Euler equations. The difference is felt by all: the mathematician sees it perhaps as a change of