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Mathematical theory of incompressible viscous fluids, by Carlo Marchioro and Mario Pulvirenti. Applied Mathematical Sciences, vol. 96, Springer-Verlag, Berlin and New York, 1994, xi+283 pp. \$49.00. ISBN 0-387-94044-8

This book is about fluids which are ideal—incompressible and inviscid. Incompressibility is the property of volume preservation: as the fluid flows, any region in it conserves exactly its volume. Inviscid fluids are free of internal friction; in particular they do not wet boundaries. Are such idealizations reasonable? Are they necessary? Or are they irrelevant? This subject is close to the heart of at least three professions: engineering, mathematics, and physics. As is often the case, practical knowledge about fluids precedes in many regards theoretical knowledge. In recent years, however, boundaries between the disciplines have become more blurred: experimental and computational advances made the mathematicians' and theoretical physicists' preoccupations closer to engineering purposes. The incompressible Euler equations describe ideal fluids. "Real" Newtonian fluids are described by the Navier-Stokes equation. The Euler equations capture the main feature of the Navier-Stokes equation—its nonlinearity. Friction is represented in the Navier-Stokes equations by a linear term, but this is the term with the highest number of derivatives. Thus the Navier-Stokes equations are a singular perturbation of the Euler equations. The difference is felt by all: the mathematician sees it perhaps as a change of type in the equation, the physicist as the introduction of a scaling relation, the engineer as the appearance of boundary layers. The Euler equations are hyperbolic, in the sense that the evolution of solutions can be determined from arbitrary (reasonable) initial data, that the time arrow does not have any intrinsic meaning for solutions, and that disturbances propagate with finite speed. The Navier-Stokes equations are parabolic: they retain the first property but not the other two. The Euler equations are invariant under a two-parameter family of rigid scale changes: one can rescale time and space independently. The Navier-Stokes equations are invariant only under rigid rescalings in which the square of length is rescaled proportionally to time. (The proportionality constant is the viscosity of the fluid.) If the fluid is in contact with walls, then the vanishing viscosity limit of Navier-Stokes equations is not, in engineering practice, the Euler equation: boundary discrepancies do not vanish.

The main mathematical questions regarding the Euler equations are: Do smooth solutions exist for all time? Or do singularities develop in finite time? Do nonsmooth but physically significant initial data lead to classical weak solutions? Do solutions of the Navier-Stokes equation converge to solutions of the Euler equations away from boundaries?

There exists a significant difference between the state of the art for the two-dimensional Euler equations and the three-dimensional equations. This difference can be understood in terms of the evolution of the vorticity. The vorticity is the antisymmetric part of the gradient of velocity. It is a scalar-valued function in two-dimensional and a vector-valued one in three-dimensional. The vorticity is advected but not stretched (the scalar function is rearranged by a time-dependent area-preserving transformation); the three-dimensional vorticity is stretched.

In two dimensions it is well known that smooth solutions exist for all time. The present estimates for two-dimensional smooth flows predict superexponential growth in time for certain quantities; it is not known if these predictions are sharp. The questions regarding weak solutions are more complicated. There are classes of initial data for which a unique global weak solution exists [1]. Among these, vortex patches have been studied extensively. Vortex patches are particular weak solutions of the two-dimensional incompressible Euler equations whose vorticity is a step function. The vortex patch boundary is an evolving, self-deforming curve in the plane, which obeys an integro-differential equation, the contour dynamics equation [2]. The area of the patch is conserved. Circles and Kirchhoff ellipses are special solutions. The circles are linearly and even mildly nonlinearly stable but strongly unstable. Motivated by numerical evidence and analogies to three-dimensional Euler vortex stretching [3], Majda [4] suggested the possibility of formation of finite time singularities in the boundary of the patch. Recently, however, Chemin ([5], see also [6]) proved that smooth boundaries stay smooth for all time.

The evolution of other nonsmooth data, such as vortex sheet initial data, is less well understood. The initial value problem is not well posed in terms of the sheets alone, and even the concept of weak solution needs perhaps to be extended.

In three dimensions (the physical case) the problem of global smooth solutions is open. There exists a connection between the behavior of the direction field of the vorticity and possible formation of finite time singularities in the Euler [7] and Navier-Stokes equations [8].

The vanishing viscosity limit has been understood only in the smooth, no-boundaries case [9].

Marchioro and Pulvirenti wrote a useful book addressed to the mathematical physics community. After a first chapter which reviews some of the basic fluid dynamics, the authors prove basic existence theorems in Chapter 2. The two-dimensional result is done using the vorticity equation; the three-dimensional one via a Galerkin method. This already exposes the reader to two of the most useful methods in the theory. Chapter 3 is devoted to the classical problem of stability of solutions; it concentrates on stability of steady solutions and explains both classical and more modern approaches. The mild nonlinear stability of circular vortex patches is proved. Chapter 4 is devoted to the point vortex system; the authors have made significant contributions to this area. Chapter 5 deals briefly with approximation methods. Chapter 6 is devoted to the evolution of discontinuities. The authors prove a Cauchy-Kovalewkaya theorem for vortex sheets and offer a discussion of water waves. The final chapter is on turbulence.

The book is pleasantly written. It touches upon remarkably many deep and important mathematical problems in fluid mechanics, and it is a very welcome addition to the, sadly, short list of mathematical treatments of fluid dynamical problems.

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