BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 3, July 1995

Analysis of and on uniformly rectifiable sets, by Guy David and Stephen Semmes. Math Surveys Monographs, vol. 38, Amer. Math. Soc., Providence, RI, 1993, xii + 356 pp., $98.00. ISBN 0-8218-1537-7

This book is a mixture of harmonic analysis and geometric measure theory. It is a quantitative study of the geometry of d-dimensional subsets of $\mathbb{R}^n$ and analysis on them, d being an integer with $0 < d < n$. "Quantitative" means that definitions, theorems, etc., come with estimates. The concept of d-dimensionality used in this book is very general; it refers only to size and has nothing to do with smoothness. To be more precise, the sets studied are those closed sets E for which there exists a constant C such that

$$(1) \qquad r^d/C \leq H^d(E \cap B(x, r)) \leq Cr^d \quad \text{for } x \in E, r > 0.$$

Here H^d is the d-dimensional Hausdorff measure, but that is not important: any measure H^d on E satisfying (1) is comparable with the d-dimensional Hausdorff measure. Such sets E are called regular. Of course, many curves and surfaces are regular, but so are many countable unions of curves and surfaces and many Cantor-type sets.

Here are some basic questions the book, together with the earlier work [DS], answers to a large extent. On what kind of regular sets are natural counterparts of classical analysis valid, such as various forms of Littlewood-Paley theory in terms of square function estimates? On what kind of regular sets does the Calderón-Zygmund theory of singular integral operators work; that is, many natural singular integral operators are bounded in $L^p(E)$-spaces? What is the right notion of quantitative rectifiability? (The qualitative rectifiability is one of the basic concepts of geometric measure theory.) There are other questions, and one might expect that the answer (if there is one) depends on the question. The surprising overall result of the book is that this is not so: the same answer is valid to almost all such questions, and that is uniform rectifiability.

Before describing the book itself, let us look at some background, beginning with singular integrals. For the classical theory and other relevant aspects of harmonic analysis, see the books of Stein [S1, S2]. Remember that we are working on lower-dimensional subsets of $\mathbb{R}^n$. A fundamental topic in this connection is the study of the Cauchy integral operator on curves of the complex plane $\mathbb{C}$. This operator is related to boundary value problems of analytic functions, partial differential equations on planar domains with minimal smoothness on the boundary, etc. The following is a basic question: for which rectifiable curves Γ in $\mathbb{C}$ is the operator

$$g \mapsto \int_\Gamma \frac{g(\zeta)}{\zeta - z} dH^1\zeta$$

bounded in $L^p(\Gamma)$ for $1 < p < \infty$? Now H^1 is just the length measure, and the integral is interpreted in the principal value sense. Based on the works of Calderón [C] and Coifman, McIntosh and Meyer [CMM], this question was answered by David [D1]. The answer is that this happens if and only if Γ is regular in the sense of (1) (in fact, only the inequality $H^1(\Gamma \cap B(x, r)) \leq Cr$ is essential, since for curves the lower bound is automatic). The same is true also for many other singular integral operators generated by kernels other than $1/z$. For d-dimensional surfaces, $d \geq 1$, such a simple characterization seems impossible, but David and Semmes have obtained many interesting results, for example, in [D2, S].

In [DS] and their book the authors study singular integrals on general d-dimensional regular sets E. They showed in [DS] that a large class of "d-dimensional" Calderón-Zygmund singular integral operators are bounded on $L^p(E)$, $1 < p < \infty$, if and only if E is uniformly rectifiable. For $d = 1$ this simply means that E is contained in a regular curve (regular in the sense of (1)) but the higher-dimensional notion is more complicated; we come to that later.

Another aspect of classical analysis is the following question on the differentiability of functions: when is a function $f \in L^2(\mathbb{R})$ almost everywhere differentiable with $f' \in L^2(\mathbb{R})$? This is essentially answered by a result of Stein and Zygmund; see [S1]: $f\colon \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous with $f' \in L^2(\mathbb{R})$ if and only if

$$\int_0^\infty \int_{-\infty}^\infty t^{-2}|f(x+t) + f(x-t) - 2f(x)|^2\, dx \frac{dt}{t} < \infty.$$

What about higher dimensions? Differentiability means approximation by affine functions; so to formulate a result in $\mathbb{R}^d$ which allows an interpretation to sets, let us introduce for a locally integrable function $f\colon \mathbb{R}^d \to \mathbb{R}$ the quantity

$$\gamma(x, t) = \inf_a t^{-d-1} \int_{B(x,t)} |f(y) - a(y)|\, dy$$

where $x \in \mathbb{R}^d$, $t > 0$, and the infimum is taken over all affine functions $a\colon \mathbb{R}^d \to \mathbb{R}$. Then $\nabla f \in L^2(\mathbb{R}^n)$ if and only if

$$\int_0^\infty \int_{\mathbb{R}^d} \gamma(x, t)^2\, dx \frac{dt}{t} < \infty \tag{2}$$

and this integral is comparable to $\int |\nabla f|^2$; see, for example, [D].

If we apply this to the graphs of Lipschitz functions, we obtain a quantitative estimate on how well Lipschitz graphs are approximated by hyperplanes at different scales. To measure such approximation for general regular d-dimensional subsets E of $\mathbb{R}^n$, set

$$\beta(x, t) = \inf_P (t^{-1} \sup\{\operatorname{dist}(y, P) \colon y \in E \cap B(x, t)\})$$

and

$$\beta_1(x, t) = \inf_P t^{-d} \int_{E\cap B(x,t)} (t^{-1}\operatorname{dist}(y, P))\, dH^d y$$

for $x \in E$, $t > 0$, where the infimum is taken over all affine d-planes P in $\mathbb{R}^n$. One way to define uniform rectifiability is via β_1: E is uniformly rectifiable if and only if there is $C < \infty$ such that

$$\int_0^R \int_{E\cap B(a,R)} \beta_1(x, t)^2\, dH^d x \frac{dt}{t} \le CR^d \tag{3}$$

for $a \in E$, $R > 0$. Comparing this with (2), one can see that the uniform rectifiability is indeed closely related to the differentiability properties of functions. If $d = 1$, β_1 can be replaced by β in (3). This is then a special case of Jones's traveling salesman result in [J]. The estimate (3) is a scale-invariant square function estimate stating that $\beta_1(x, t)^2 dH^d \times t^{-1}\, dt$ is a Carleson measure on $E \times R$. Various other estimates of this type, both geometric and analytic, are studied in the book and shown to be equivalent to (3).

In addition to the above two aspects of the interplay between classical analysis and the geometry of regular sets, many others have been studied in the monograph. Let us now see how the book relates to geometric measure theory and in particular to d-dimensional rectifiable sets in $\mathbb{R}^n$. Their theory was developed by Besicovitch in the 1920s and 1930s for $d = 1$, $n = 2$ and by Federer in the 1940s for general d and n; see, e.g., [F, Fe, M]. Here we are studying sets E with $H^d(E) < \infty$ without any quantitative estimates on the size. Among them rectifiable sets form essentially the largest class where many basic geometric properties of smooth surfaces have reasonable analogues. Such properties are, for example, existence of tangent planes (defined in a measure-theoretic, approximate, way), parametrization by Lipschitz maps (or rather covering with Lipschitz surfaces), and the analogue of Lebesgue's density point theorem. All these are qualitative, without any estimates. The uniformly rectifiable sets provide an interesting and useful quantitative theory of rectifiability. For example, the authors prove a characterization of uniform rectifiability in terms of the three above-mentioned concepts: approximation by planes, parametrization by Lipschitz maps, and density ratios. For the first, this is given by (3).

The notion and the theory of uniform rectifiability are relatively new. They stem from the work of David and of Semmes on the singular integrals and other types of analysis on surfaces. The first systematic exposition was given in [DS]. The present monograph extends [DS] considerably. Practically all the results proved here are due to the authors, and most of them are new.

This is not an easy book to read; the arguments are often technically quite involved. But the authors use a lot of effort in trying to explain the underlying ideas and basic concepts. In particular, the long introductory Part I of about fifty pages is excellent. There background is illuminated from various points of view, and a summary of the main results with helpful explanations is given. After reading Part I, one already has a pretty good picture of what uniform rectifiability is all about. Although the book answers many questions, it also asks many; it will quite likely be a fruitful source for future research. All in all, reading this book is a very rewarding experience, and I recommend it to all those interested in analysis and geometry and their interplay.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 3, July 1995

Free Lie algebras, by Christopher Reutenauer. Clarendon Press, Oxford, 1993, xvii+269 pp., $86.00. ISBN 0-19-853679-8

A *free Lie algebra* is a Lie algebra freely generated by a set called *alphabet*. The theory of free Lie algebras grew from the need of computing formally with commutators $[a, b] = ab - ba$ (using only the Jacobi identity and antisymmetry). It can be traced back to the turn of the century in the work of Campbell (1898), Poincaré (1899), Baker (1904), and Hausdorff (1906).

Given a set X (alphabet) and a ring K of coefficients, we can look at the immediate "structural relatives" of a free Lie algebra which are

(i) The free Lie algebra $L(X)$ (or $L_K(X)$)
(ii) The free magma $M(X)$
(iii) The free group $F(X)$
(iv) The algebra of noncommutative polynomials $K\langle X\rangle$
(v) The algebra of noncommutative formal power series $K\langle\langle X\rangle\rangle$