that refine the π_n. We have (all indices have weight n)

$$\Gamma_n = (\oplus_\lambda K.\pi_\lambda) \oplus (\oplus_{\lambda>\mu}\pi_\mu\Gamma_n\pi_\lambda)$$

where $\geq$ is an order on *partitions* (decreasing compositions) such that $\pi_\mu\pi_\lambda \neq 0 \Rightarrow \lambda \geq \mu$. Note that in [1] another realization of the algebra $\oplus\Sigma_n$ is given in terms of *noncommutative symmetric functions.*

In closing, one considers an algebra larger than the algebra of symmetric power series. This is the algebra of *quasisymmetric functions* over a totally ordered alphabet $(X, <)$. This algebra is generated by the *monomial quasisymmetric functions*

$$M_C = \sum_{x_1<x_2<\cdots<x_k} x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$$

where $C = (i_1, i_2, \ldots, i_k)$ is a composition. It is true that $M_{C_1}M_{C_2} = \sum M_{C_i}$, so that the monomial functions form a linear basis of the algebra (in fact, this algebra can be shown to be the Grothendieck ring of a suitable category, the product corresponding to the tensor product). The ring of quasisymmetric functions is dual to the descent algebra by means of the formula

$$F_{C(\sigma)}(XY) = \sum_{\sigma=\beta\alpha} F_{C(\alpha)}(X)F_{C(\beta)}(Y),$$

where XY is the lexicographic product of the totally ordered sets $(X, <)$, $(Y, <)$, and $C(\sigma)$ is the integer vector of the "jumps" of the descent set of σ. These quasisymmetric functions are used to give generating series of special sets of permutations.

The book by Christopher Reutenauer is very well written: proofs are as elementary as they can be, and examples are given to illustrate the notions that are introduced progressively and at the right moment. We are indebted to him for having put some order in the welter of properties and connections surrounding the free Lie algebras. The book is recommended for everyone, whether familiar or less than familiar with the subject, for pleasure and for knowledge.

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Morse homology, by M. Schwarz. Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel and Boston, MA, 1993, ix+235 pp., $49.50. ISBN 3-7643-2904-1

A Morse function f on a manifold M is a function which is bounded from below, which is proper and whose critical points are non-degenerate. Morse theory studies the questions, what a manifold knows about the critical points of a Morse function and what the Morse function knows about the manifold. For an excellent survey the reader may consult [1].

Taking a Riemannian metric on M, we can build the gradient ∇f of f and study the associated negative gradient flow

$$\dot{x} = -\nabla f(x).$$

The first important observation is that due to the properness of f this differential equation generates a semi-flow in forward time, which we denote by $M \times \mathbb{R}^+ \to M \colon (x, t) \to x \cdot t$. The second observation is that given a flow line $x(t) = x \cdot t$, the map $t \to f(x(t))$ is strictly decreasing iff $x(0)$ is not a critical point of f. This implies quite easily that f^a is a deformation retract of f^b provided there is no critical level contained in (a, b). Here f^c denotes the compact subset of M consisting of all $x \in M$ with $f(x) \leq c$.

Hence the homotopy type of f^b changes only at critical levels of f. One can show that if c is the only critical level in $[c-\varepsilon, c+\varepsilon]$, then $f^{c+\varepsilon}$ is obtained from $f^{c-\varepsilon}$ by attaching a cell for every critical point on level c. The dimension of the cell is precisely the Morse index of the critical point.

Using the standard properties of homology, one can use the above fact to derive the Morse inequalities. The Morse inequalities give a lower bound for the number of critical points of a specific Morse index in terms of the Betti numbers of the manifold. A fancy way to formulate them is the following. Denote by C_k the free Abelian group generated by the critical points of Morse index k and by C_* the direct sum of the C_k. A sequence of maps $\partial_k \colon C_k \to C_{k-1}$ induces a map $\partial \colon C_* \to C_*$ of degree -1. The Morse inequalities say that there exists a map ∂ as above such that

$$\partial^2 = 0 \quad \text{and } \operatorname{kern}(\partial)/\operatorname{image}(\partial) = H_*(M).$$

The following question arises: Can we construct, given the Morse function f, the homology in a natural way? The answer is yes, and the first steps towards an intrinsically defined Morse homology were undertaken by Thom, Smale and Milnor [15, 14, 17, 16, 18]. The crucial point in recovering the homology is to understand the attaching map to $f^{c-\varepsilon}$ when one passes the critical level c. The ideas of Thom, Smale and Milnor were taken up by Franks [12], who showed that the attaching data is encoded in the following information. Assume x and y are critical points for f with Morse index difference $\mu(y) - \mu(x) = 1$. Given a generic metric g on M, the stable manifold $W_s(y)$ and the unstable manifold $W_u(x)$ intersect transversally and their intersection is one-dimensional (the importance of the transversality in this respect was recognized by Smale). This set intersects the level $f^{-1}(c-\varepsilon)$ transversally. The number of intersection points in the above intersection turns out to be finite. Alternatively, we could remark that we have a free **R**-action defined by the flow, so that $(W_u(x) \cap W_s(y))/\mathbf{R}$ consists of a finite number of points. With the help of some orientation conventions their number can be algebraically counted. This number $n(x, y)$ carries all information one needs to know about the attaching map if we want to study the change of homology.

All this was rediscovered by Witten in his important paper [19]. This paper together with Gromov's seminal work in [13] and Conley and Zehnder's proof of one of the Arnold conjectures [4] was the starting point of Floer's homology theory [8, 10, 5, 7, 11, 9, 6].

Coming back to our original question, it turns out that the homology can be constructed in a natural way from the Morse function. Let C_k be as described above. We define $\partial: C_k \to C_{k-1}$ as follows

$$\partial x = \Sigma_{\mu(y)=k-1}\#(x, y)y,$$

where $\#(x, y)$ is the algebraic count of connecting trajectories between x and y. Observe that $\mu(x) - \mu(y) = 1$, so that the number of trajectories between x and y is finite. If one prefers to work with coefficients in $\mathbf{Z}_2$ instead of with coefficients in $\mathbf{Z}$, the so-called algebraic count of trajectories is the number of trajectories modulo 2. The crucial point is that $\partial^2 = 0$. This can be seen as follows. We have

$$\partial^2(x) = \Sigma_{\mu(z)=k-2}(\Sigma_{\mu(y)=k-1}\#(x, y)\#(y, z))z = \Sigma_{\mu(z)=k-2}\alpha(x, z)z.$$

We observe that $\alpha(x, y)$ is the algebraic number of broken trajectories from x to z. For simplicity let us take $\mathbf{Z}_2$ coefficients. The central fact is now that via a compactness and an implicit function type argument we can establish a bijective correspondence between the broken trajectories and the ends of the one-dimensional manifold $(W_u(x) \cap W_s(z))/\mathbf{R}$. This is a manifold without boundary; hence it consists of a disjoint union of intervals and circles. The decisive argument is that the number of ends is even; hence 0 modulo 2.

Another important aspect in studying flows is the continuation principle. This was exhibited already in a very clear form by Conley in his seminal work in [3]. In our case we can describe it as follows. Let f_0 and f_1 be two different Morse functions, and let g_0 and g_1 be two Riemannian metrics. We take an arc of smooth functions $(F_s)_{s\in\mathbf{R}}$ such that

$$F_s = f_0 \quad \text{for all } s \leq -s_0 \quad \text{and} \quad F_s = f_1 \quad \text{for all } s \geq s_0.$$

Moreover we take an arc of Riemannian metrics interpolating between g_0 and g_1. We study the parameter-depending flow equation

$$\dot{x}(s) = \nabla_s F_s(x(s)).$$

Here $\nabla_s F_s$ denotes the gradient of F_s with respect to the metric g_s. We look for solutions of the above equation connecting a critical point x_- of f_0 for $s \to -\infty$ to a critical point x_+ of f_1 for $s \to +\infty$. Both critical points are assumed to have the same Morse index. Again the solutions can be counted and used to define a chain map Φ by

$$\Phi: C(f_0, g_0) \to C(f_1, g_1): \Phi(x) = \Sigma_{\mu(y)=\mu(x)}\#(x, y)y,$$

where x is a critical point of f_0 and the sum is taken over critical points y of f_1. It turns out that for a generic interpolation between our data the induced map in homology is independent of the choices involved, and we obtain what is called a connected simple system, namely, a functorial construction which associates to every generic pair (f, g) a homology group and to two pairs (f_0, g_0) and (f_1, g_1) a uniquely determined isomorphism between the associated homology groups.

That these homology groups are isomorphic to the singular groups can be shown by Witten's Hodge argument (see the appendix in [2]) or—even simpler—by Floer's argument involving the Conley index [11].

In essence, what is crucial about all this is that the singular homology of a manifold can be computed from a Morse function by considering only the connecting orbits between critical points of Morse index difference 1 and 2. Moreover we do not need the whole gradient flow. This observation is of vital importance for Floer's homology theory. He concludes that under favorable circumstances, studying an infinite-dimensional problem, we do not need a well-defined Morse index, only a well-defined difference of Morse indices. Moreover we do not need a flow, only a good notion of a connecting orbit. Floer implemented his breakthrough ideas in the study of the symplectic action functional and in the study of the Chern-Simons functional. Both are variational problems on infinite-dimensional spaces having infinite classical Morse indices. But it turns out that the difference of the Morse indices makes sense, not necessarily as an integer but as a cyclic number in some $\mathbf{Z}_N$, depending on the situation.

In his book Schwarz develops Floer theory in finite dimensions. The book begins with a very nice introduction which describes some of the historic developments and the main principles.

In the second chapter the spaces of connecting trajectories between critical points are studied in great detail. It is shown that these spaces can be viewed as the zero sets of nonlinear Fredholm operators. Another important ingredient which is studied with great care is the associated transversality theory. For the property $\partial^2 = 0$ the compactness properties of the orbit spaces are proved as well as the particular form of the implicit function theorem which is needed. This chapter about the trajectory spaces forms the technical heart of the theory. Though Schwarz concentrates on finite-dimensional Morse theory, this chapter is written in such a way that it can be used as a very comprehensive guideline in what to do if the trajectory spaces turn out to be solutions of partial differential equations rather than ordinary differential equations.

The next chapter is concerned with orientation questions for the spaces of trajectories. In the finite-dimensional theory it is related to orienting the unstable manifolds. However, if we work in an infinite-dimensional setting, the finite-dimensional ideas do not work anymore, and we need a new orientation concept. We have to orient the determinant bundles of families of linear Fredholm operators in a way compatible with the gluing construction in the chapter about the trajectory spaces. In an appendix Schwarz shows that in the finite-dimensional case the naive orientation concept is equivalent to the fancy "Fredholm orientation" concept.

In Chapter 4 the Morse homology is developed. That is, the boundary operator and the chain homotopies are constructed. A very interesting part is devoted to the construction of a relative homology group as well as the construction of an induced homomorphism by a smooth map. Schwarz then shows that the theory satisfies the Eilenberg-Steenrod axioms. In Chapter 5 some extensions are given. How a Morse cohomology can be defined and how Poincaré duality and the cup product fit into the picture are described.

Summing up, this is a very useful book explaining the ideas and the analytical difficulties of Floer theory in the finite-dimensional case. The proofs are written with great care, and Schwarz motivates all ideas with great skill. We not only learn something about Floer's seminal ideas but also gain a new insight into finite-dimensional Morse theory. This is an excellent book.

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K-theory for real C^*-algebras and applications, by Herbert Schröder. Pitman Research Notes in Mathematics, vol. 290, Longman Scientific & Technical, Harlow, Essex, and Wiley, New York, 1993, xiv+162 pp., \$49.95. ISBN 0-582-21929-9

One of the most important developments in mathematics over the last few decades has been the emergence of deep connections between the subjects of