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*K-theory for real $C^*$-algebras and applications*, by Herbert Schröder. Pitman Research Notes in Mathematics, vol. 290, Longman Scientific & Technical, Harlow, Essex, and Wiley, New York, 1993, xiv+162 pp., $49.95. ISBN 0-582-21929-9

One of the most important developments in mathematics over the last few decades has been the emergence of deep connections between the subjects of operator algebras and geometry/topology. This interplay has had great significance for both areas, and a contemporary specialist in either field is almost compelled to have a good basic working knowledge of the other.

A pervasive idea of great importance throughout mathematics is to study a "space" by means of the "functions" on the space. Interpreted literally, this idea lies at the heart of such subjects as algebraic geometry and probability theory as well as classical functional analysis; interpreted slightly more broadly, it includes such subjects as the representation theory of groups, and in recent years it has given powerful new ways to describe and study singular "spaces" such as the leaf space of a foliation, where the only naturally defined nonpathological object is a (noncommutative) algebra of "functions". (See the introduction of [Co] for a detailed and highly recommended survey of the theory of singular spaces and noncommutative "function algebras".)

To avoid getting too far afield with generalities, we will take as our model the well-known procedure of associating to a compact Hausdorff space $X$ the algebra $C(X)$ of real- or complex-valued continuous functions on $X$. Continuous maps between spaces correspond (contravariantly) to homomorphisms between the function algebras. Since $X$ can be recovered from $C(X)$ as the space of maximal ideals with the hull-kernel topology (or as the set of algebra homomorphisms to the base field with the topology of pointwise convergence), the entire topological structure of $X$ is encoded in the algebraic structure of $C(X)$. So at least in principle, every aspect of the theory of compact Hausdorff spaces can be studied by means of the associated algebras. In practice, of course, this is not the right way to look at most things in topology.

A pessimist or cynic might say the algebra approach is not the right way to look at any aspect of topology, but this is not so! The most important example is topological $K$-theory, the study of vector bundles. Vector bundles over a compact Hausdorff space $X$ correspond naturally to finitely generated projective modules over $C(X)$ (the module corresponding to a bundle is the space of continuous sections), and it turns out that the most elegant and in a certain sense the most natural way to develop all the powerful machinery of $K$-theory (it is a periodic extraordinary cohomology theory) is through the Banach algebra structure of the $C(X)$'s and their matrix algebras. This approach was pioneered by Atiyah, Karoubi, Swan, and Wood, among others; a detailed account can be found in [Kr]. Surprisingly, and notably, the theory and even the proofs largely carry over verbatim to noncommutative Banach algebras and give a theory lying at the heart of the modern subject of "noncommutative topology". For the $K$-theory of Banach algebras, see [Bl], or, for a more detailed and up-to-date treatment [WO]. The algebra approach to topological $K$-theory has also developed in a different direction into the important subject of algebraic $K$-theory; see [Ro] for an excellent modern account.

The reader will notice that I have been vague about the specification of the base field in the above description. This was deliberate, and the distinction between the real and complex theories leads to the essence of the book under review. $X$ can be recovered from real $C(X)$ or complex $C(X)$ in an identical way, so the base field does not matter here; the projective modules over real (resp. complex) $C(X)$ correspond to real (resp. complex) vector bundles over $X$, and topological $K$-theory includes the distinct, but parallel, real and complex $K$-theories (as well as quaternionic $K$-theory).

People like me who study operators and Banach algebras have a strong prejudice for working over the complex numbers, since the theory of complex Banach algebras is in many respects simpler and better behaved than the theory of real Banach algebras. Some of the differences are merely due to the fact that $\mathbf{C}$ is algebraically closed, others to the beauty of the theory of complex analytic functions, and still others to more subtle or arcane differences between $\mathbf{R}$ and $\mathbf{C}$ such as the structure of Clifford algebras. In many of the appearances of operators and operator algebras in applications such as mathematical physics, arguments are given to justify the desirability or necessity of working over $\mathbf{C}$ and interpretations (such as phase) given to the imaginary components; but some of these arguments seem suspiciously like rationalization for the mathematical convenience of working in complex spaces. Experience has shown that some applications of operator algebras and noncommutative topology to geometry/topology, at least, require considering real operator algebras, and with sufficient care most of the important facts about the $K$-theory of complex operator algebras have nice real analogs. This is the theory developed in Schröder's book.

Before describing the real theory, let us go into more detail about some of the many developments in (complex) $K$-theory and noncommutative topology. Very early, the close connections between $K$-theory and the index theory of elliptic differential operators were noticed. Indeed, the work of Atiyah and Singer on their celebrated index theorem was one of the first uses of $K$-theory (work of Atiyah and Hirzebruch a few years earlier marked the beginning of $K$-theory as a branch of algebraic topology; the first notions of $K$-theory go back to Grothendieck's work on the Riemann-Roch theorem). Atiyah proposed a definition of $K$-homology in terms of generalized elliptic operators, a theory completed and expanded by the work of Brown-Douglas-Fillmore and Kasparov on extensions of $C^*$-algebras. Kasparov then used a similar approach beginning with the idea of an elliptic operator to develop a very general and powerful bivariant theory, called $KK$-theory, including both $K$-theory and $K$-homology. Closely related ideas involving summable Fredholm modules and smooth structure led Connes to develop cyclic cohomology, the beginning of noncommutative differential geometry [Co]. Work has progressed in several directions, from the algebraic approaches to $K$-theory and cyclic cohomology of Cuntz, Quillen, Tsygan, and many others, to applications of noncommutative topology/geometry to topology problems such as the Novikov Conjecture and existence of metrics of positive scalar curvature on manifolds. $K$-theory has also become a vital tool within the field of operator algebras, from the study of subfactors and connections to knot theory and mathematical physics, initiated by Jones, to Elliott's classification program for separable nuclear $C^*$-algebras.

Schröder's book concentrates on only one aspect of the theory and its applications, namely, generalized index theorems for families of elliptic operators. The original Atiyah-Singer index theorem asserts that, under suitable hypotheses, the Fredholm index of a (complex) elliptic operator on a manifold can be computed from topological data from the manifold and the symbol of the operator. In many instances of interest, one deals with a family of elliptic operators parametrized by a "space" $Y$. The general principle is that to such a family an index can be associated which is not a number but an element of the $K$-theory $K^0(Y)$. (In this picture, the usual Fredholm index is not a number but an element of $K^0(pt) \cong \mathbf{Z}$, or more properly of the $K$-theory of the compact operators.) In some cases, the "space" parametrizing the family is a singular space, and the index takes values in the $K$-theory of the associated noncommutative algebra of "functions"; for example, a pseudodifferential operator on a foliated manifold which is elliptic in the leaf direction may be thought of as such a family parametrized by the leaf space, and its index lies in the $K$-theory of the foliation $C^*$-algebra. (The equivariant version of the Atiyah-Singer theorem can also be thought of in a similar way, with the "space" being the group dual.) There are then a host of generalizations of the original Atiyah-Singer Index Theorem, giving index formulas from topological data.

Atiyah and Singer also considered the case of real elliptic operators. In contrast to the complex case, where the topological index data can be (and originally were) phrased in cohomological terms, one cannot avoid the use of $K$-theory in the real index theorem. Also, in the case of a family of real elliptic operators, the index takes values in the $K$-theory of a commutative real $C^*$-algebra which is not necessarily of the form real $C(X)$ for a compact Hausdorff space $X$. Despite the obvious possibility, and desirability, of obtaining real analogs of the various generalizations of the (complex) Atiyah-Singer results, the only instance of the substantial use of real $C^*$-algebras and $K$-theory in a closely related area seems to have been the work of Rosenberg et al. on the Novikov conjecture and the positive scalar curvature problem, where topological results are obtained which apparently cannot be established by the complex theory. (Hitchin did use the real Atiyah-Singer index theorem for families to obtain a real longitudinal index theorem for a fibration.) Another real index theorem has finally appeared in the book under review.

The first two chapters of this book give a description of real $C^*$-algebras and $K$-theory and $KK$-theory in the real case. The discussion of real $C^*$-algebras, while not as complete as that in [Go], contains some topics not readily available elsewhere, such as the theory of real crossed products, and also a detailed study of the real analogs of such standard (complex) $C^*$-algebras as the Toeplitz algebra, Bunce-Deddens algebras, irrational rotation algebras, and Cuntz and Cuntz-Krieger algebras. A section on real Clifford algebras is of particular interest, since they hold the key to much of $K$-theory including Bott periodicity, and are just complicated enough that they cannot be hidden as is usually done in the complex case. The development of real $K$-theory holds no surprises, although the approach is a bit unconventional and notationally nicely facilitates Bott periodicity and certain index calculations; this approach holds merit also in the complex case. Real analogs of the main results of $K$-theory for complex $C^*$-algebras are proved, including versions of the Pimsner-Voiculescu exact sequence and Connes' Thom isomorphism. Real $K$-theory has period 8, of course.

The treatment of $KK$-theory in Chapter 2 gives a nice summary of the theory, including the equivariant case, but really contains nothing unavailable in previous expositions such as Kasparov's original papers or [Bl] (which remains the only comprehensive treatment of $K$-theory for operator algebras, although sadly out of date and out of print only eight years after publication). Kasparov's formidable machinery works equally well in the complex, real, and Real cases (a Real $C^*$-algebra is the complexification of a real $C^*$-algebra); the theories are formally identical, with the substantive differences buried in such details as the structure of graded tensor products. The reviewer's prejudices showed up in [Bl], where the real and Real cases were scarcely mentioned, but the statements and proofs of results in the general case are for the most part identical to those in [Bl] (in fact, Schröder frequently simply refers to [Bl] for proofs).

Chapter 3 contains the treatment of real index theorems. After a discussion of orientation in the real setting, the Atiyah-Singer index theorem for families of real elliptic operators is set up and proved in the more modern language of correspondences. Then the main new result of the book, the longitudinal index theorem for foliations in the real case, is proved. With the machinery already set up, the theorem and proof are formally nearly identical to the complex theorem of Connes and Skandalis. The last section contains some applications to existence of metrics on a foliated manifold whose scalar curvature is strictly positive in the leaf direction.

The exposition in this book is quite complete in terms of the development of the ideas and statements of the results, but it is rather short on proofs in some sections. In many instances where proofs are identical to ones previously published, a reference is simply given. Unfortunately, some of these references are to manuscripts, such as the author's dissertation, which are not widely available. Also, in a number of instances a result is asserted for real $C^*$-algebras, but the proof referred to is for complex $C^*$-algebras, leaving it to the reader to check that the proof also works in the real case. (One hopes the author has carefully checked that this is indeed true!) In some other cases, where a proof is similar but not identical to a published one, the author resorts to the annoying but sometimes necessary practice of referring to the original proof with just a description of the items to be changed. The result is that, while one does not need to be completely familiar with the complex results to follow the book, any reader other than a few experts (or a casual reader content to merely browse through the statements of results) will have to have an array of reference materials available for constant use. It must be said that, given that it was not feasible to include complete proofs of all results, the author has exercised generally good judgment in deciding what to include and what to omit.

This book will be of interest to specialists in either operator algebras or global analysis, although the operator algebra prerequisites may be a challenge for some global analysts. While the importance to anyone interested in index theory should be obvious, the main benefit to an operator algebra specialist will probably be to present further evidence that the interest and importance of operator algebras is not limited to the complex case.

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*Hilbert space operators in quantum physics*, by Jiří Blank, Pavel Exner, and Miloslav Havlíček. American Institute of Physics Translation Series in Computational & Applied Mathematical Physics, American Institute Physics, New York, 1994, xiii+593 pp., $75.00. ISBN 1-56396-142-3

Quantum mechanics is not so much a physical theory as a framework for physical theories. In one context it describes electrons and nuclei obeying nonrelativistic dynamics and interacting via electrostatic forces. This is already enough to give a reasonable description of the properties of ordinary matter. Somewhat closer to the frontier of physical theory are relativistic quantum fields that describe quarks (which make up the particles that form the nuclei) and leptons (including electrons). These interact through the medium of other relativistic quantum fields, the gauge fields. An ultimate theory of nature may extend quantum mechanics to incorporate gravity or some yet unknown physical effect. However, in every case, whatever the particles and whatever the forces, quantum mechanics is supposed to provide the framework.

Must this be so? The physicist Steven Weinberg poses this question in his recent popular book [1]. He considers the validity of quantum mechanics as subject to experimental test, but he speculates that "quantum mechanics may survive not merely as an approximation to a deeper truth... but as a precisely valid feature of the final theory."

Quantum mechanics is formulated in abstract mathematics, and there seems to be no simple pictorial explanation of quantum concepts. If quantum mechanics is ultimate truth, then this is good news for mathematicians—we can deal with abstraction—but bad news for almost everyone else.

The mathematics is that of Hilbert space. A *Hilbert space* is a complex vector space with an inner product that makes it into a complete metric space. Every closed subspace $M$ of a Hilbert space $\mathscr{H}$ is also a Hilbert space, and its orthogonal complement $M^\perp$ is also a closed subspace. The projection theorem says that for every vector $\psi$ in the Hilbert space and for every closed subspace $M$, there is an orthogonal projection of $\psi$ onto $M$. This is the unique vector $\psi'$ such that $\psi = \psi' + \psi''$ with $\psi'$ in $M$ and $\psi''$ in $M^\perp$. The *projection operator* $E_M$ onto $M$ is defined by $E_M\psi = \psi'$. The projection operator $E_{M^\perp}$ onto the orthogonal complement $M^\perp$ is then given by $E_{M^\perp}\psi = \psi''$.

In quantum mechanics the *state* of a system is determined by a unit vector $\psi$ in a Hilbert space $\mathscr{H}$. Each *quantum event* corresponds to a closed subspace $M$ of the Hilbert space $\mathscr{H}$. The fundamental postulate of quantum mechanics is