REFERENCES

1. Elie Cartan, *Sur les domaines bornés homogénes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 116–162.
2. Gerald B. Folland, *Spherical harmonic expansion of the Poisson-Szegö kernel for the ball*, Proc. Amer. Math. Soc. **47** (1975), 401–408.
3. H. Fürstenberg, *A Poisson formula for semisimple Lie groups*, Ann. of Math. **77** (1963), 335–386.
4. Sigurdur Helgason, *Groups and geometric analysis*, Academic Press, New York, 1984.
5. J. E. Littlewood, *On functions subharmonic in a circle*. III, Proc. London Math. Soc. **32** (1931), 222–234.
6. Walter Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.
7. David Ullrich, *Radial limits of* $\mathcal{M}$*-subharmonic functions*, Trans. Amer. Math. Soc. **292** (1985), 501–518.

WALTER RUDIN
UNIVERSITY OF WISCONSIN-MADISON

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The general topology of dynamical systems, by Ethan Akin. Graduate Studies in Mathematics, vol. 1, American Mathematical Society, Providence, RI, 1993, x + 261 pp., $50.00. ISBN 0-8218-3800-8

Topological dynamics, which is the subject of the book under review, is a subfield of what might be called "abstract dynamics". The latter covers a vast canvas and for this reason is difficult to define precisely. One possible (although incomplete) definition is the asymptotic, or long-term, properties of families (usually groups or semigroups) of self maps of spaces. It includes *topological dynamics* (continuous maps of topological spaces), *ergodic theory* (measure-preserving transformations of probability spaces), and *smooth dynamics* (diffeomorphisms of smooth manifolds). Of course there are many interrelations among them, as well as with other branches of mathematics. The subject of dynamics dates back (at least) to Newton and the modern, abstract approach to Poincaré. The abstract point of view was expounded more explicitly by Birkhoff, who considered groups of transformations on metric spaces. A concise historical survey appears in the introduction to the book by Furstenberg [F].

The book of Nemytskii and Stepanov [NS], to which the author draws attention in the introduction, was probably the first to present systematically the properties of solutions of differential equations in terms of flows (actions of the reals) on metric spaces. The abstract theory was laid out in detail in the Colloquium volume of Gottschalk and Hedlund [GH], in which arbitrary group actions were emphasized. The subject then took on a life of its own and dealt with topics rather far from its origins. For instance, the books of Ellis [E] and the reviewer [A] concentrate on the structure of minimal flows, and the specialized books of Furstenberg [F] and Glasner [G] develop connections with number theory and group theory, respectively.

Akin's book consists largely of topological dynamics, but most of the topics studied are those which are motivated by smooth dynamics. As such it constitutes a bridge between the two areas. Indeed, in the preface the author says that his aim is to describe "what every dynamicist should know ... from topological dynamics." Much of the book consists of the author's presentation, in a purely topological setting, of theory developed by Conley and Smale. Thus Akin's book brings topological dynamics back to its origins (with due attention to developments of recent years).

Rather than maps, the framework of the book, as set out in the first chapter, is that of relations on compact metric spaces and the generation of new relations from old ones. Recall that a relation on a set X is a subset f of $X \times X$, and if $x \in X$, $f(x) = [y|(x, y) \in f]$. If g is another relation on X, then $g \circ f = [(x, z)|(x, y) \in f, (y, z) \in g$ for some $y]$. For a relation f the orbit relation is $\mathscr{O}f = \cup_{n=1}^{\infty} f^n$ (where $f^2 = f \circ f$ and f^n is defined inductively). $\mathscr{O}f$ is the smallest transitive relation containing f. If X is a topological space, f is a closed relation if f is a closed set. Of course, a continuous map on a compact metric space is a closed relation. Given a relation f, the smallest closed transitive relation $\mathscr{G}f$ containing f can be generated, starting with f, by alternately closing and "transitizing" (in general transfinitely often) the relations obtained.

Two relations of dynamical interest are Ωf and ωf, defined, respectively, by $y \in \Omega f(x)$ if and only if there are sequences $x_n \to x$, $y_n \to y$, $k_n \to \infty$ with $y_n \in f^{k_n}(x_n)$, and $y \in \omega f(x)$ if and only if there is a sequence $y_n \in f^{k_n}(x)$ with $k_n \to \infty$. (The latter set is, in the case of maps, the *omega limit set* of x.) If f is a continuous map, then x is a *nonwandering point* for f if and only if $x \in \Omega f(x)$, and x is a *recurrent point* if and only if $x \in \omega f(x)$. In general, if g is a relation, the *cyclic set* of g is $|g| = [x \in X|x \in g(x)]$, and various notions of recurrence can be defined by the condition that $x \in |g|$.

In particular, chain recurrence can be approached from this point of view, and this is one of the author's most successful accomplishments. The chain relation is defined by $\mathscr{C}f = \cap\{\mathscr{O}\{\overline{V_\epsilon} \circ f|\epsilon > 0\}$ (where V_ϵ denotes the ϵ neighborhood of the diagonal). If f is a continuous map, $y \in \mathscr{C}f(x)$ ("x chains to y") if and only if for every $\epsilon > 0$ there are $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$ and $d(f(x_{i-1}), x_i) < \epsilon$ for $i = 1, \ldots, n$. The relation $\mathscr{C}f$ is transitive and closed and therefore contains $\mathscr{G}f$ (properly, in general), and x is chain recurrent if $x \in |\mathscr{C}f|$. Interesting relations between attractors and chain recurrence are developed. (A closed set A with $f(A) = A$ is an *attractor* if there is a neighborhood U of A such that $\Omega f(x) \subset A$ for every $x \in U$; a *repellor* is an attractor for f^{-1}.) For maps, several useful equivalent conditions of apparent varying strengths are given. One of these is that an attractor is determined by its "trace" on the chain recurrent set, in the sense that $A = \mathscr{C}f(A \cap |\mathscr{C}f|)$. Also, the chain relation can be recovered from the attractor structure: if x is a chain recurrent point, then $y \in \mathscr{C}f(x)$ if and only if x and y belong to the same attractors.

If f is a homeomorphism, the chain recurrent set can also be constructed from the limit point set $l(f) = \overline{\alpha f(X) \cup \omega f(X)}$ (where $\alpha f(x)$ denotes the *alpha limit set* of x: $\alpha f = \omega f^{-1}$). A closed f invariant set F is called $l(f)$ *separating* if $F \cap l(f)$ is open and closed in $l(f)$. In this case $\omega f(x) \subset F$ if and only if $\omega f(x) \cap F \neq \varnothing$; the set of all such x is called the *inset* $W^+(F)$, and the *outset* $W^-(F)$ is defined similarly (using $\alpha f(x)$). A generalization of a lemma of Shub and Nitecki says: *Let* F *be an* $l(f)$ *separating set, and suppose* $x \notin W^+(F)$. *Then* (a) : $\Omega f(x) \cap F \neq \varnothing$ *implies* $\Omega f(x) \cap (W^+(F) \setminus W^-(F)) \neq \varnothing$, *and* (b) : *the same as* (a) *with* $\Omega f(x)$ *replaced by* $\Omega\mathscr{C}f(x)$. Just a few words regarding the proof, which is quite intricate: it is relatively easy, using standard techniques, to find points y_1, $y_2 \in \Omega f(x)$ (or $\Omega\mathscr{C}f(x)$), each satisfying "half" of what is required—that is, $y_1 \notin W^-(F)$ and $y_2 \in W^+(F)$. What is difficult is to find a point with both of these properties. For this, an interesting combinatorial lemma concerning pairs of subsets of $\mathbb{Z}$ (or $\mathbb{R}$, which allows the Shub-Nitecki lemma to be proved for real actions) is employed.

This lemma is used to study invariant decompositions. An invariant decomposition $\mathscr{F}$ for f is a pairwise disjoint finite family of closed invariant sets whose union contains $l(f)$. Thus each $F \in \mathscr{F}$ is $l(f)$ separating. Regarding $\mathscr{F}$ as a discrete metric space, five relations of different strengths are defined on $\mathscr{F}$. These include $\mathscr{F}_1 = [(F_1, F_2)|W^-(F_1) \cap W^+(F_2) \neq \varnothing]$, $\mathscr{F}_4 = [(F_1, F_2)|\Omega f(F_1) \cap F_2 \neq \varnothing]$ and $\mathscr{F}_5 = [(F_1, F_2)|\mathscr{C}f(F_1) \cap F_2 \neq \varnothing]$. The Shub-Nitecki lemma is used to show that the transitive extensions (denoted $\mathscr{O}(\mathscr{F}f)$) of all these relations agree. (This turns out to be equivalent to $\mathscr{F}_5 f \subset \mathscr{O}(\mathscr{F}_1 f)$.) This result is used, in turn, to obtain certain "attractor-repellor pairs" for f, the so-called attractor structure of the invariant decomposition $\mathscr{F}$, and then to obtain information on the location of the basic sets for f (the equivalence classes on $|\mathscr{C}f|$ of $\mathscr{C}f \cap \mathscr{C}f^{-1}$).

Here is another consequence of the Shub-Nitecki lemma (communicated by Moe Hirsch). Consider a (real) flow on a compact metric space with a finite fixed point set F such that every semitrajectory converges to some point of F. Then every chain recurrent set belongs to a cycle of orbits connecting fixed points (and therefore the chain recurrent set coincides with F if and only if there are no such cycles).

The final chapter, "Hyperbolic Sets and Axiom A Homeomorphisms", defines these notions topologically and develops their topological properties. If f is a homeomorphism, a closed invariant subset K of X is called *isolated* if K has a neighborhood U such that every invariant subset of U lies in K. (U is called an isolating neighborhood for K.) An attractor or repellor for f is an isolated invariant set, and a basic set for f is isolated if and only if it is an open and closed subset of $|\mathscr{C}f|$. A homeomorphism f is *expansive* if the diagonal is isolated in $X \times X$. (Equivalently, an orbit in $X \times X$ which is sufficiently close to the diagonal is contained in the diagonal.) A closed invariant subset K of X is called expansive if there is a closed neighborhood U of K such that if K_0 is the maximum invariant subset of U, then the restriction f_{K_0} is expansive. (Note that this says more than f_K is expansive.)

If $\{x_n\}$ and $\{y_n\}$ are (finite or infinite) sequences in X and $\gamma > 0$, we say that $\{x_n\}$ γ *shadows* $\{y_n\}$ if $d(x_n, y_n) \leq \gamma$ for all n. (So, for example, the expansive condition says that there is a $\gamma > 0$ such that if the orbit $\{f^n(x_1)\}$ γ shadows $\{f^n(x_2)\}$, then $x_1 = x_2$.) Now let f be a homeomorphism, and let $A \subset X$. Then A is said to satisfy the shadowing property if for every $\epsilon > 0$ there is a $\delta > 0$ such that every (finite) δ chain in A can be ϵ shadowed by a 0 chain in X. That is, every δ chain in A can be ϵ chained by a piece of an orbit. A closed invariant subset K of X is called *topologically hyperbolic* if K is expansive and satisfies the shadowing property.

The main result on topological hyperbolicity involves a weakening of the isolation condition. Assume that the topologically hyperbolic set K is isolated "rel Per f". This means that there is a neighborhood U of K such that any periodic orbit which is contained in U is in fact contained in K (so an isolated invariant set is isolated rel Per f). Then the chain recurrent set $|\mathscr{C} f_K|$ consists of finitely many basic sets, each of which is an isolated invariant set on which f is topologically transitive. Also, the periodic points are dense in $|\mathscr{C} f|$.

The purely topological point of view is abandoned in the last part of this last chapter. It is now assumed that f is a C^1 diffeomorphism of a smooth Banach manifold X. Hyperbolic fixed points and compact invariant sets are defined as usual (in terms of splittings of the tangent bundle), and it is shown that a hyperbolic invariant set for a diffeomorphism is topologically hyperbolic. Similar results are proved for Axiom A diffeomorphisms (defined as those for which the chain recurrent set is hyperbolic). On the way some standard smooth results are established (Hartman's theorem and the Shadowing Lemma). The proofs in this section rely on results—some of them quite difficult—from the previous chapter on "Fixed Points".

This book is a substantial contribution. It is remarkable how Akin has been able to distill the purely topological aspects of smooth dynamics. For me, the most successful aspect of the book is its treatment of chain recurrence and related notions such as attractors and basic sets. (The treatment of chain recurrence extends over several chapters and contains much more than I have summarized above.) This is where the author's "relationspeak" really pays off. The results on topological hyperbolicity do capture the topological properties of hyperbolic systems, but of course not the quantitative properties, as indicated, for example, by Lyapunov exponents. Moreover, without the assumption of smoothness and the resulting property of transversality, only very weak perturbation results can be obtained (as the author points out).

The somewhat uncompromising use of relations, even when what is mainly of interest is maps, allows a great deal of information to be packed in. This comes at a price. This is not a book one reads casually or just picks up to find a result. The difficulties are exacerbated by a tendency towards extremely long statements of theorems (some extending for more than a page). Moreover, sometimes definitions, as well as frequently used notation, are introduced as part of the statements of theorems.

References and attributions for specific results are almost nonexistent. (There is a page of "Historical Remarks" at the end of the book, but these are not very specific.) It seems likely that many of the theorems as presented are due to the author. One such is the Shub-Nitecki lemma, to which Akin contributed the most difficult part. Another occurs in a chapter on "Invariant Measures for Mappings": Let f be a topologically transitive map of X, and let $\mathrm{In}_P(f)$ denote the set of f invariant Borel probability measures on X. For $x \in X$ let $M(x)$ denote the collection of measures which arise from x as weak limits of ergodic averages. Let $Con(f)$ be the set of x such that $M(x)$ consists of a single measure (so the ergodic averages converge). $Con(f)$ is of full measure (with respect to every f invariant measure), and there is a nonempty closed connected set I^* of $\mathrm{In}_P(f)$ such that $M^{-1}(I^*) = [x|M(x) = I^*]$ is residual. When I^* contains more than one measure (for example, when f is minimal and not uniquely ergodic), then $Con(f) \cap M^{-1}(I^*) = \varnothing$. So in this case $Con(f)$ is of first category and $M^{-1}(I^*)$ has invariant measure zero.

While the phenomenon of residual sets and sets of full measure being disjoint is well known, this dynamical formulation is quite natural and appears to be new. (However, this is an example of what was alluded to above—it is necessary to search through earlier theorems just to understand the notation.)

It is a standard practice for a reviewer to regret the absence of certain topics, and this will be no exception. The most striking omission is topological entropy (surely something that "every dynamicist should know"), which would have fit in well with the discussion of decompositions. A discussion of the specification property and the construction of Markov partitions would also have been of interest. There is a brief introduction to symbolic dynamics, but it is not related to the other topics in the book. (On the other hand, symbolic dynamics is a world in itself, and a more extended treatment would have lengthened the book significantly.)

This monograph is Volume 1 in a new AMS series, Graduate Studies in Mathematics. It is natural to ask whether it is appropriate for such a series. Well, in one sense it certainly is. The prerequisites are modest (mostly general topology of metric spaces), and the proofs are clear and well motivated. Another attractive feature is the extensive collection of exercises, of varying levels of difficulty; in some cases, they develop non-trivial extensions of the theory. On the other hand, a student learning dynamics may wish at first to concentrate on works which are more explicitly in smooth dynamics. These include the book of Mañé [M] and the forthcoming books of Katok and Hasselblatt [KH] and Clark Robinson [R]. Another useful reference, especially for topics related to hyperbolicity, is the book of Shub [S]. Be that as it may, Akin's interesting and well-written book is a valuable addition to the literature and should be in the library of every dynamicist.

References

[A] J. Auslander, *Minimal flows and their extensions*, North-Holland, Amsterdam, 1988.

[E] R. Ellis, *Lectures in topological dynamics*, Benjamin, New York, 1969.

[F] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, Princeton, NJ, 1981.

[G] S. Glasner, *Proximal flows*, Lecture Notes in Mathematics, vol. 517, Springer-Verlag, Berlin and New York, 1976.

[GH] W.H. Gottschalk and G.A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, RI, 1955.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, London and New York, 1995.

[M] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag, Berlin and New York, 1983.

[NS] V.V. Nemytskii and V.V. Stepanov, *Qualitative theory of differential equations*, Princeton Univ. Press, Princeton, NJ, 1960.

[R] C. Robinson, *Dynamical systems: Stability, symbolic dynamics, and chaos*, CRC Press, Boca Raton, FL, 1994.

[S] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, Berlin and New York, 1987.

JOSEPH AUSLANDER
UNIVERSITY OF MARYLAND
E-mail address: jna@math.umd.edu