the predecessor volume [6] at hand. Many further results are given as exercises, with solutions or references.

Besides, I found no serious lacuna in the index and very few misprints in the book, none of them bothersome. (At the author's request, I mention that the remark on page 274 should be deleted.) As in [5 and 6], for which Helgason received the 1988 Steele Prize for expository writing, the style is very fluent and pleasant, conducting the reader at a regular pace. I think the present book will be a most valuable (and reasonably priced) reference for anyone interested in Radon transforms and analysis on semisimple Lie groups.

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Traveling wave solutions of parabolic systems, by Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert. Transl. Math. Monographs, vol. 140, Amer. Math. Soc., Providence, RI, 1994, xii + 448 pp., \$142.00. ISBN 0-8218-4609-4

The theory of systems of nonlinear parabolic P.D.E. is a centerpiece of modern applied mathematics, and such equations have a virtually ubiquitous presence as mathematical models in science and engineering. Such systems often assume the form

$$u_t = D\Delta u + f(u, \nabla u),$$

where $u \in \mathbb{R}^n$ and D is an appropriate matrix. When the spatial variable x is one-dimensional and unrestricted, the equations frequently admit travelling waves, which are special translation invariant solutions of the form $u(x, t) = U(x-ct)$, where $U = U(\xi)$ is the profile of the wave which propagates through the one-dimensional spatial domain at constant velocity c. Such solutions are studied as a paradigm for the behavior exhibited in many model problems, and in certain situations they play a key role in the characterization of the behavior of more general solutions of the initial value problem.

Two of the most frequently encountered classes of equations are reaction-diffusion systems wherein $f = f(u)$ is semilinear, and systems of viscous conservation laws wherein $f = f(u)_x$. The book under review is concerned exclusively with wave phenomena in reaction-diffusion systems. Viscous systems of conservation laws are more readily understood as perturbations of hyperbolic systems, and the theory has a rich relationship to the theory of hyperbolic waves. However, much of the existential theory associated with travelling involves general methods that are applicable to both areas. This is not the case in regard to the stability analysis of waves, and the set of techniques that have been most successful with viscous shock profiles are distinct from those which have been developed in connection with reaction-diffusion problems; see e.g. Liu [18].

The first instances in which travelling wave solutions of reaction-diffusion systems were investigated were in the celebrated papers of Kolmogorov, Petrovskii, and Piskunov [17] and Fisher [9], in which u is a scalar and $f = u(1-u)$. Since then, and particularly over the last twenty-five years, the investigation of travelling wave phenomena has played a central role in the qualitative theory of solutions of reaction-diffusion systems.

One aspect of this particular class of solutions is that the profile U satisfies an associated system of O.D.E.'s,

$$\begin{aligned} U' &= V \\ DV' &= -cV + f(U), \end{aligned} \tag{1}$$

together with appropriate conditions at $\pm\infty$, and so their analysis straddles partial differential equations and dynamical systems and frequently draws upon methods from both areas. When the dimension of u is two or more, the dimension of (1) is at least four, and the location of solutions with certain prescribed behavior at infinity, such as heteroclinic or homoclinic orbits, can be a difficult task. Several distinct points of view have evolved concerning both the existence and the asymptotic stability of travelling waves, and they have each produced their own corpus in the literature. Below, I discuss several of the main themes that have emerged together with my opinions regarding their strengths and limitations. It is not possible to give a complete account of this work here, and the outline below omits much important work.

Degree theoretic methods. The application of Leray-Schauder degree to travelling waves is not straightforward because of the lack of compactness of the associated operators due to the unbounded spatial domain. This difficulty can be addressed either by working in suitably weighted function spaces or by truncating the spatial domain to a finite interval and by passing to a limit as the size of the domain becomes infinite. The book under review is principally concerned with developing the Leray-Schauder method for travelling waves, both in an abstract setting and for specific systems. The authors' use of weighted function spaces provides a particularly appealing and clean manner in which to develop the theory. The authors have made key contributions in this regard (see e.g. [25, 26]), and much of the text is concerned with their work in this area.

An important advantage of this method is that it is particularly well suited to the study of multidimensional waves in cylindrical domains (see e.g. [2]). There is no inherent limitation in regard to the extent of its applications. However, the results that have been produced so far, and, in particular, those presented in this book, have required some sort of monotonicity both in the nonlinearity and in the profile itself in order to obtain the a priori estimates required in the abstract formulation. This is usually equivalent to the presumption that the system in question admits a comparision principle, a hypothesis which is not satisfied in general. It places a rather severe constraint on the type of system and on the type of solution to which the Volperts' main existence and stability theorem can be applied.

The Conley Index and related methods. In his monograph on the index theory for isolated invariant sets of flows, Conley [3] set forth a general program for measuring Morse decompositions in terms of a new topological invariant. The principal illustrative examples in the early applications of the Conley index, much of which was joint work of Conley and Smoller, were connecting orbit problems derived from the travelling wave equations (1) for parabolic systems; see [24] for an account of this work and additional references. Since then there has been continuing research, both generally and specifically, on the application of these methods to the existential theory of travelling waves. This work has been carried forward by several investigators, including Mishaikow, Reinek, Franzosa, and McCord, who have developed and applied several of Conley's ideas about the connection matrix and connected simple systems, which are refinements of the Conley homotopy index (see e.g. [19]). The results of Mishaikow and Hutson [20] on cooperative species in mathematical ecology have substantial overlap with the results on monotone systems due to the Volperts.

There are other closely related approaches, such as shooting methods and Wazewski's principle, which have also proved to be effective tools in many situations; see, for example, Hastings [11].

These methods are extremely general and apply in principle to the location of any bounded solution of (1), including heteroclinic, homoclinic, and periodic solutions. The difficulty is always in the construction of a suitably refined isolating region in the phase space of (1). Results obtained from crude neighborhoods supply scant information about the qualitative characteristics of the wave; finer constructions typically provide more information but depend quite delicately upon the form of the nonlinearities in the equations.

Classical and geometric singular perturbation theory. A frequently studied problem is the singular limit in which some of the diffusion coefficients in the matrix D tend to zero. Typically, the wave is resolved into distinct layers consisting of slowly modulated outer segments which are separated by rapid, inner transition layers. There has been a great deal of formal work in the form of matched asymptotic expansions of such solutions (see e.g. Fife [6, 8]); occasionally it has been possible to translate these calculations into rigorous results by means of the implicit function theorem.

Heteroclinic and homoclinic waves can also be viewed geometrically as the intersection of stable and unstable manifolds of rest points. This observation is most useful when the intersection is transverse, so that it persists under small perturbations. In the setting of singular perturbation problems, it is frequently possible to describe the behavior of invariant manifolds of the relevant rest points in the singular limit, and when the limiting manifolds intersect transversely, a rigorous result for the perturbed problem can be obtained. The global invariant manifold theorems of Fenichel [5] have provided the key geometric machinery in this regard, and this has been an area of much recent activity (see e.g. Jones and Kopell [14]). There have also been some interesting connections with other problems in dynamical systems, such as the existence of multi-pulse solutions of Hamiltonian systems.

The classical and geometric theories are obviously limited to certain ranges of parameters; however, these are often the regions of parameter space where interesting behavior occurs. The construction of one-dimensional waves also has important applications in the setting of singular limit problems to the tracking of the evolution of multi-dimensional interfaces, wherein matched asymptotic expansions of the evolving interface are obtained by composing the one-dimensional wave with the solution of a geometric equation for the free boundary in the asymptotic limit. Such results are for the most part only available as formal calculations, except in the case of a scalar equation. Here, because of the maximum principle and other special features of scalar equations, a relatively complete rigorous theory is available.

Stability and bifurcation of solutions. The stability of wave solutions of the bistable scalar equation was solved in the important paper of Fife and MacLeod [7] which appeared in 1977. This result made essential use of the maximum principle, and its generalizations are limited systems which admit some kind of comparison principle. At about the same time, some general abstract theorems on nonlinear stability based upon semigroup theory due to Sattinger [23] and Henry [12] appeared which required the underlying wave to be linearly stable. However, checking the theorem's key hypothesis seemed to severely limit its application in many situations, since the associated linearized operator L about a travelling wave $U(\xi)$

$$Lp = Dp'' + cp' + df(U(\xi))p$$

has variable coefficients and in general is not self-adjoint.

Since then several new ideas have emerged for studying the spectrum of linear operators arising in this manner which have permitted the application of the general theorems of [23, 12] to many interesting systems. One key innovation was work by Evans [4] on the stability of pulses in neurophysiology, in which he defined a certain analytic function $D(\lambda)$ whose roots coincide precisely with the eigenvalues of L. The Evans function was subsequently used by Jones [13] in a rigorous proof of the stability of the pulse solution of the FitzHugh-Nagumo system. Jones's proof was based on geometric arguments; a more analytical approach to the same problem but also based on the Evans function was given by Yanagida [27]. The Evans function has also been used in the study of the stability of wave solutions of other classes of equations such as the generalized KdV equation; see Pego and Weinstein [22].

The Evans function together with certain key geometric ideas introduced by Jones were further developed and generalized by Alexander, Gardner, and Jones [1]. Specifically, a stability index was devised which relates the multiplicity of eigenvalues of the operator L interior to some fixed curve K in the spectral plane to the first Chern number of a certain complex vector bundle over S^2. This has made available a collection of geometric and topological tools in the stability analysis of waves, which have been particularly useful in the setting of singular limit problems; see e.g. Gardner and Jones [10].

The SLEP method ("singular limit eigenvalue problem") due to Nishiura and Fujii [21] is another important advance in the stability analysis of solutions of singularly perturbed systems of two equations. Their approach, which is based upon the classical theory of singular perturbations and matched asymptotic expansions, has led to some beautiful and deep rigorous results on the stability and bifurcation of waves in the singular limit [16].

In Sattinger's paper, stability with respect to perturbations with exponential spatial decay is proved in certain situations in which the linearization has continous spectrum passing through the origin. In a recent advance, Kapitula has developed an interesting and quite subtle theory of nonlinear stability relative to spaces of perturbations with algebraic decay [15]. Kapitula's theorem reveals the corresponding temporal decay rates that can be expected in this situation, which are also algebraic.

I will conclude with a description of the material in the book under review. After a leisurely introduction, the book is divided into three parts. Part I presents general results of a global nature, beginning with a complete account of travelling wave solutions of a single equation. The theory of Leray-Schauder degree for travelling waves in an abstract setting is presented, and the theory is then applied in the proof of an existence theorem for monotone systems. Next, the authors present results on the structure of the spectrum of elliptic linear operators on cylindrical domains and apply them to the linearization of certain scalar equations. The Sattinger/Henry theorem on nonlinear stability of travelling waves in one space variable is presented, and an application is given to the stability analysis of monotone travelling wave solutions of locally monotone systems. In so doing, the authors present a new and important proof of the simplicity of the zero eigenvalue of the linearization about such a solution.

The second part presents the theory of Hopf bifurcation of waves on cylindrical domains and the exchange of stabilities. The usual infinite-dimensional version of the Hopf bifurcation theorem does not apply to travelling waves due to the presence of the translational eigenvalue at the origin for all parameter values. Nevertheless, the authors are able to prove that Hopf bifurcation occurs in this setting. I am not aware of such a result elsewhere in the literature.

In Part III, the authors use the theory developed in the first two parts in the analysis of waves in several physical applications. In particular, they apply their theorems on the existence and stability of monotone waves to systems arising in the theory of chemical reactors and also to combustion with complex chemistry. They conclude with some interesting estimates on the wave velocities and a description of bifurcation phenomena in combustion theory. The results in this part will be of great interest to applied mathematicians and scientists working in these areas.

The book is a well-written and welcome addition to the literature on this subject. The subject matter is focused upon the research interests of the authors, and much of the text presents the results of papers that are only available in Russian and are not easily accessible to Western readers. Most of the book would be suitable for graduate students after a course in P.D.E., and the first part of the book contains an excellent introduction to the elementary aspects of the subject. The book does not convey to the reader an impression of the breadth and mathematical richness that have been the hallmark of this large and still developing field, and it is unfortunate that the authors did not make an effort to give a more balanced presentation, at least to the extent of providing an expository discussion of some of the other viewpoints mentioned above. A neophyte would need to consult other references to obtain a balanced introduction to the whole field. However, the authors do provide an extensive bibliography, and their book will be a valuable resource for researchers who are already working in this area.

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Normal forms and bifurcations of planar vector fields, by S.-N. Chow, C. Li, and D. Wang. Cambridge University Press, Cambridge and New York, 1994, viii + 472 pp., $69.95. ISBN 0-521-37226-7

This book studies planar vector fields. The phase portraits of such fields contain a kind of concise, non–verbose information similar to e.g. city–maps. Vector fields show up as dynamical systems, which in general need not have a 2–dimensional phase space. However, there are many situations that allow reduction to this special case of dimension 2.

A familiar reduction principle, e.g., known from physics, uses symmetry. An example within reach of the techniques of the book is the following. Consider a periodically forced oscillator of the form

$$\ddot{x} = f(x, \dot{x}, t), \quad x \in \mathbb{R}, \; t \in \mathbb{R},$$

say with period 1 in the time t, of which we like to study resonances. A corresponding dynamical system is

$$\dot{x} = y, \; \dot{y} = f(x, y, t), \; \dot{t} = 1,$$

which has a 3–dimensional phase space $\{x, y, t\}$. One way to deal with this is to look at its Poincaré or stroboscopic map P, which is again 2–dimensional. By definition it follows the associated 3–dimensional flow from $((x, y), 0)$ to $(P(x, y), 1)$, so from the section $t = 0$ to $t = 1$. Thus a fixed point of P corresponds to a periodic solution of period 1, etc. Now a general 2–dimensional map can still be quite a complicated object, but fortunately the present problem is to study the Poincaré map in the neighborhood of such a fixed point, say p. If S denotes the semisimple part of the derivative D_pP,