BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 4, October 1995

Notions of convexity, by Lars Hörmander. Progress in Mathematics, vol. 127, Birkhäuser, Boston, 1994, viii+414 pp., \$49.50. ISBN 0-8176-3799-0

This is an excellent exposition on the notions of "convexity in a wide sense", with a strong emphasis on their application to the theory of linear partial differential equations and complex analysis. "Convexity in a wide sense" includes subharmonic functions (Chapter III), plurisubharmonic functions, Siamese twins of pseudo-convex sets (Chapter IV), convexity with respect to differential operators (the so-called P-convexity) (Chapter VI), etc. Some topics one might expect from the title (e.g. the role of convexity in functional analysis) are not included. (Concerning functional analysis, however, it should be noted that Chapter II, particularly Section 2.1, is an excellent introduction to the subject; although materials are confined to finite-dimensional vector spaces, the author seems always to have in mind the counterpart in the infinite-dimensional case.) Subjects that are relevant to concrete problems in complex analysis and the theory of linear partial differential operators seem to have been given top priority, and I think this selection of topics is reasonable and appropriate. Actually by this selection this book has become a kind of reliable and exhaustive reference book to the celebrated monographs of the author: [CASV] *An introduction to complex analysis in several variables*, North-Holland, 1990, 3rd edition; and [ALPDO] *The analysis of linear partial differential operators*, I–IV, Springer, 1983–1985. (Here and in what follows the same symbol as in this book is used to designate a reference when it is possible.) When I was a student, the most standard reference book on subharmonic functions was the book by Radó, which actually I was acquainted with through [CASV]; I am sure this book will become a standard reference concerning subharmonic and plurisubharmonic functions. One charming feature that makes this book something beyond a reference book is that even classical and traditional subjects are treated in a way reflecting the ideas and thinking of the author, who has been contributing to the development of the theory of differential equations in a substantial manner during the past four decades; typical examples can be easily found, e.g., in Sections 1.6, 2.1, 2.3, 3.1, 3.3, 4.4, 4.6, etc. (more specifically, e.g., Theorems 2.3.7, 3.1.14, 3.3.23, 3.3.26, etc.). Needless to say, Section 4.2 ("Existence Theorems in L^2 Spaces with Weights") and Section 6.1 ("P-Convexity") are a kind of résumé of the celebrated achievements of the author himself. It is also a pleasure to those who have learned much from the author, as I have, to find materials which remained unpublished for several decades

(Section 2.5 and Chapter V). Coming to this rather personal point, let me mention an anecdote which may be of some interest in connection with Section 6.3 and Chapter VII. Almost three decades ago, H. Komatsu, interviewed by some "writers" of a coterie journal, including me, emphasized the importance of studying partial differential equations on a complex domain, pointing out the difficulty in solving the simplest equation, $\partial u/\partial z_1 = f$, globally (Sûgaku no Ayumi **13** (1967), 83). Apparently not only his enthusiasm for developing the hyperfunction theory, which is closely tied up with analysis on a complex domain, but also his stay in Stanford, from where he had just returned to Tokyo and where he had been influenced by Hörmander, the author of this book, motivated it. Thus the global solvability of the equation $\partial u/\partial z_1 = f$ became a fashionable subject among those interested in hyperfunction theory; an interesting counterexample was immediately found by I. Wakabayashi (Proc. Japan Acad. **44** (1968), 820–822), and then followed a decisive result by H. Suzuki [1] (cf. Lemma 6.3.1 of this book), which, in the case of first-order operators on $\mathbb{R}^2$, Suzuki related to the renowned Nirenberg-Treves condition for the local solvability of first-order linear differential equations through the geometric study of the interplay of the characteristic curves and the real domain (J. Math. Soc. Japan **23** (1971), 18–26). Although these results of Suzuki were interesting and instructive, the development of the theory of boundary value problems for (overdetermined) elliptic systems by Kashiwara and the reviewer was necessary for obtaining the result in full generality; this is the main theme of Chapter VII, which is based on the thesis by J. M. Trépreau (1984). To be more specific, in Chapter VII the fact that the Nirenberg-Treves condition (Ψ) is equivalent to microlocal surjectivity at a characteristic point for a microdifferential operator of principal type is obtained (Theorem 7.4.7) by studying when

$$\partial/\partial z_1\colon \mathscr{A}_{z_0}(X) \to \mathscr{A}_{z_0}(X)$$

is surjective, where X is an open subset of $\mathbb{C}^n$, z_0 is a point in the boundary ∂X, ∂X is smooth and strictly pseudo-convex at z_0, and $\mathscr{A}_{z_0}(X)$ denotes the germ at z_0 of functions analytic in X (Theorem 7.2.3 and Theorem 7.3.5.). In this almost final part of the book, the reader encounters again the notion of quasi-convexity and its applications given at the beginning of the book (Definition 1.6.3 and Section 1.7, which is, together with the extension to subharmonic functions (the final part of Section 3.2, i.e., p.167–p.171), due to Trépreau [2] and Ancona [1]. See also their plurisubharmonic version (Theorem 4.1.30 and Corollary 4.1.31), which is important in the proof of Theorem 7.2.3). To draw the conclusions from the results on the surjectivity of $\partial/\partial z_1 : \mathscr{A}_{z_0}(X) \to \mathscr{A}_{z_0}(X)$, the author uses two basic results in microlocal analysis: a result which allows us to play the game of microlocal analysis on the conormal bundle of ∂X (Theorem 4 of Kashiwara-Kawai [1]) and an equivalence result for microdifferential operators of principal type with the same principal symbol (Theorem 2.1.2 of Sato-Kawai-Kashiwara [1], Chapter II).

I would like to end this review by expressing my personal wish: I hope students in complex analysis read this book to find that the theory of partial differential equations is closely related to the theory of analytic functions of several complex variables, and still more, I wish they would try to attack this adjacent field. [CASV] is an excellent book also for this purpose, but this new book is probably better suited for this purpose, as it contains much more material which is peculiar to the theory of partial differential equations. As one of the old students of [CASV], I believe the new generation will benefit much from this new book by Hörmander.

TAKAHIRO KAWAI
RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 4, October 1995

Thermomechanics of evolving phase boundaries in the plane, by Morton E. Gurtin. Oxford University Press, Oxford and New York, 1993, xi+148 pp., \$54.00. ISBN 019-853694-1

The theory of solidification and crystal growth is mathematically complex and technologically vital. It is based on partial differential equations and numerical computation, with geometric measure theory, differential geometry and calculus of variations playing useful roles as well. Every industry from semiconductors to steel must predict and control solidification processes, providing a wide range of applications for mathematical modelling and numerical analysis.

Gurtin's book provides clear, thorough derivations of basic results in the mathematical modelling of solidification processes, with only elementary real analysis and thermodynamics as prerequisites. More advanced results would require substantial background in PDE, numerical or asymptotic analysis, so the scope of the book is well chosen. References are given to further work in PDE related to solidification but not to numerical or asymptotic analysis.

Specific quantitative models of solidification come in three flavors: sharp interface, phase field and lattice models. We briefly compare the three classes of models, though Gurtin discusses only the first.

Sharp interface models consist of equations of motion for the solid-liquid interface, which is idealized as a curve or surface of zero thickness. The equations of motion involve the geometry and thermodynamics of the interface and the *phases* (solid or liquid) on each side. Laplace and Young studied surface tension in liquid drops with sharp interfaces in 1805, creating a theory reformulated by Gauss twenty-five years later. It was 1892, however, before Gibbs constructed the general thermodynamics of sharp interfaces which forms one of the main threads of the book.

Phase field models combine two ideas. First, the solid-liquid interface is smeared into a smooth transition zone. In 1830, Laplace's protégé Poisson studied such a smooth transition of density across a liquid surface, which had been neglected in the Laplace-Young theory. About 1892, Rayleigh and van der Waals constructed a complete general theory of interfaces with nonzero thickness, reformulated by Cahn and Hilliard [3] in 1958. Second, a "phase field" or "order parameter" indicating smooth changes from solid to liquid is added to the thermodynamic variables, a device popular in statistical physics