But the aim of the authors is a different one, as indicated in the title of the book. Hence its middle part (Chapter 3–Chapter 6) gives a dense presentation, construction and discussion of the properties of the fundamental solution and of the Green functions for L under various boundary conditions. One can find the whole classical material, including simple and double-layer potentials, jump relations and Levi's Parametrix method. The other possibility for constructing the Green function, using the existence theory for the initial boundary value problem for $Lu = f$, is indicated in the third part of Chapter 6.

The last three chapters are devoted to the construction and discussion of the Green functions G for $L - I$. The key tool for the construction of G as a series of terms, each of which solves a Volterra equation, is the definition and study of a certain scale of Banach-spaces containing these kernel functions. This scale is defined by means of fifteen(!) (semi-)norms. Chapter 8 contains the construction of G, and properties and estimates of G are discussed in Chapter 9.

Due to the subject, the contents of the book are highly technical—a research note in the best sense. It has the further merit of containing the classical material for linear parabolic equations with variable coefficients for all types of boundary values in a form which allows a citation.

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Elliptic differential equations (*theory and numerical treatment*), by W. Hackbusch (translated from the German by Regine Fadiman & Patrick D. F. Ion). Springer-Verlag, Berlin, Heidelberg, and New York, 1992, xiv+311 pp., \$89.00. ISBN 3-540-54822-X

This book gives a rigorous account of the theory of elliptic partial differential equations and numerical approximations. There are some gaps in the sense of missing material that one would have thought would be in a book like this. However, the material that is covered is for the most part very well done. A lot of attention and care has been given to both proofs and examples, and the result is a highly readable volume that is suitable for advanced undergraduates and beginning graduate students.

The first three chapters deal with scalar second-order elliptic equations. The approach is classical with an emphasis on the mean value property, related integral representation, and maximum principles. The topology used is defined by the sup norm, and various properties of classical solutions are established. These include existence, uniqueness, and continuous dependence on data.

The treatment of boundaries is particularly distinguished, not only for the theory developed but also for the examples supplied which illustrate the limitations of the theory. To cite but one example, let Ω be a bounded region in $\mathbb{R}^n$ with boundary Γ, and let u be a classical solution of

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \Gamma, \tag{1}$$

i.e., u is in $C^2(\Omega) \cap C^0(\overline{\Omega})$. Suppose $\Omega_n \subset \Omega$ with $\operatorname{dist}(\Gamma, \Gamma_n) \to 0$ as $n \to \infty$. It is then shown that the solution of

$$\Delta u_n = 0 \quad \text{in } \Omega, \qquad u_n = \phi_n \quad \text{on } \Gamma \tag{2}$$

converges uniformly to u in the sense

$$\lim_{n\to\infty} \sup\{|u_n(x) - u(x)| : x \in \Omega_n \cap \Omega\} = 0 \tag{3}$$

provided

$$\lim_{n\to\infty} \sup_{x\in\Gamma} \inf_{y\in\Gamma_n} \{|\phi(x) - \phi_n(y)|\} = 0. \tag{4}$$

It is natural to ask whether it is necessary to assume the existence of the solution u to (1). Can it be reduced from $\Gamma_n \to \Gamma$ and $\phi_n \to \phi$ (in the sense of (4))? The author answers this negatively with the following very nice example:

$$\Omega_n = \left\{ (x_{1,2}) \in \mathbb{R}^2 : \frac{1}{n} < x_1^2 + x_2^2 < 1 \right\},$$

$$\phi_n = \begin{cases} 0 & \text{if } x_1^2 + x_2^2 = 1, \\ 1 & \text{if } x_1^2 + x_2^2 = \frac{1}{n^2}, \end{cases}$$

$$\phi = \begin{cases} 0 & \text{if } x_1^2 + x_2^2 = 1, \\ 1 & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

This satisfies all the stated conditions (except for the existence of a solution to (1)), yet the solution of (2), which is

$$u_n(x_1 x_2) = \log(\sqrt{x_1^2 + {}_2^2}) / \log\left(\frac{1}{n}\right),$$

is not Cauchy in the sup norm topology. This theorem/example motif can be found throughout the book, and it is one of its strong points.

Finite difference approximations are taken up in Chapter 4. The treatment is compatible with the first three chapters in the sense it is classical and retains the sup norm topology. The discrete analogs of the maximum principles are reflected in various properties of the finite difference operator L_h which approximates the second-order elliptic operator L (which in this chapter is the Laplacian). These include semidiagonal dominance and nonnegativity of L_h^{-1}. The latter is developed and used to analyze the fundamental questions relating to convergence and stability. Again the treatment of boundaries is particularly distinguished.

These results are generalized in Chapter 5 to the case of arbitrary second-order scalar elliptic equations. There is also a brief (four-page) and somewhat inadequate discussion of the fourth-order case. Because of the heavy dependence on maximum principles, there is not sufficient mathematical machinery available to treat adequately anything other than the second-order case. An error estimate is given for the biharmonic equation, but as the author notes it is not best possible and predicts a slower convergence than what actually occurs.

In Chapter 6 the book shifts from a classical to a modern mode using functional analysis and Sobolev spaces. This chapter contains a complete and readable survey of the results that will be needed later for variational methods. The latter is taken up in Chapter 7 for the general $2m$th order scalar elliptic equation. This is followed in Chapter 8 with the development of finite element theory. This material can be found in scores of other references; however, the treatment in this book is concise and very well done.

By contrast, Chapter 9 contains a unique and interesting account of regularity. The application of these ideas to develop improved error estimates for finite difference approximations contains a lot of original material that cannot be found elsewhere.

The treatment of eigenvalue problems in Chapter 11 is equally strong. The approach has a number of original features, and it is without a doubt the best overall account of the subject to be found in the literature.

Chapter 12 deals with the Stokes equations and mixed finite element methods. The treatment is rigorous and reasonably complete, but the overall development seems far more complicated and technical than it need be. Research during the last few years has led to theories which contain the material in Chapter 12 as a special case but which are cleaner and more concise. The centerpiece of this work is the Agmon-Douglis-Nirenberg theory of elliptic systems. One particularly frustrating feature of this chapter is that the Agmon-Douglis-Nirenberg condition is cited but not used in any meaningful way.

Chapter 10 is the only really weak chapter in the book. It is a brief (eight-page) and rather superficial account of singular perturbation problems and equations with discontinuous coefficients. The book would actually have been strengthened if this chapter had been omitted.

Overall, this book gets high marks in spite of a few weak spots. Specialists will seek it out for the original material it contains (particularly in Chapters 9 and 11). It also can claim serious attention as a textbook for a course on elliptic partial differential equations.

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