BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 3, July 1996

Basic ergodic theory, by M. G. Nadkarni, Texts and Readings in Mathematics, vol. 6, Hindustan Book Agency, 1995, viii + 179 pp., \$28.00, ISBN 81-85931-07-0

Regarded as a part of mathematics, ergodic theory in its simplest version is the study of measure-preserving transformations of probability spaces (example: on the unit circle C in the complex plane, with normalized arclength, multiplication by a fixed c of modulus 1, which—to be interesting—should not be a root of unity. "Interesting" here means "ergodic": every measurable invariant set has measure zero or one.) A slightly less simple version is the study of one-parameter groups of such transformations, called "flows" (example: translation on the 2-torus $C >< C$ by the 1-parameter subgroup of pairs (z, cz); again, for ergodicity, c should not be a root of unity). So one is looking at probability-preserving actions of Z or R, or perhaps of the semigroups $Z+$ or $R+$. One may introduce further structure on the space and correspondingly on the transformations; the space may, for example, be a smooth or analytic manifold or a compact metric space. The subject also extends to include actions of more general groups and semigroups. Ergodic theory has important links with physical science (notably celestial mechanics and statistical physics) and engineering (notably communication theory), and its results and ideas have contributed significantly to them, as well as to other parts of mathematics itself. The most comprehensive source now available in English is I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic theory* (New York and Berlin: Springer-Verlag, 1982).

Let us for a moment discard the measure and just look at automorphisms of Borel spaces (a Borel space is a set equipped with a sigma-algebra of subsets). In the case of interest, namely, when the probability space is not too big, the Borel space may be chosen to be "standard". We remind the reader that a standard space is one which is isomorphic to a Borel subset B of a Polish space equipped with its sigma-algebra of Borel sets. It can be shown that for an uncountable B all these Borel spaces are isomorphic. Furthermore, G. Mackey and A. Ramsay proved that given a countable group of automorphisms of a standard Borel space, there is a Polish topology with the same Borel sets such that the automorphisms become homeomorphisms. This has been improved to cover appropriate actions of continuous groups, and the topology may furthermore be made compact.

In recent years there has been some interest and success in developing analogues to the ideas and results of ergodic theory, both at the level of automorphisms of standard spaces and at the level of homeomorphisms of Polish or compact metric spaces; in the latter situation, the role of sets of measure zero is sometimes played by meager sets.

The present book contains an unusual but not unreasonable combination of material. Firstly, it contains about one third to one half of a standard course in (measure-theoretic) ergodic theory, up to and including the definition of entropy. Proofs are clean and clear. Some of the less standard topics in this part of the book are:

1991 *Mathematics Subject Classification.* Primary 60XX; Secondary 04XX.

1. Kakutani equivalence.
2. Dye's Theorem: for any two ergodic measure-preserving transformations on nonatomic probability spaces there is an isomorphism between the two probability spaces carrying orbits onto orbits.
3. E. Hopf's theorem: a necessary and sufficient condition for a measure which is quasi-invariant under a transformation to be equivalent to an invariant probability measure is that the transformation cannot, in a certain measure-theoretic sense, compress the space (such a transformation is now usually called "of finite type", after Murray-von Neumann and Krieger).
4. The Ambrose-Kakutani Theorem: for every ergodic flow on a nonatomic probability space there is a measurable set intersecting almost every orbit in a discrete set.

But secondly and mainly, the book shows how analogous results, and more refined results, can be obtained at the level of automorphisms of standard spaces or homeomorphisms of compact metric spaces. Indeed, its title might more appropriately be something like *Ergodic theory and Borel automorphisms*. Thus we are shown (in some order) Borel or compact metric versions of Poincaré Recurrence, Rokhlin's lemma on periodic approximation, induced transformations and Kakutani equivalence, Rokhlin's countable generator theorem, Dye's Theorem, the Hopf theorem of no. 3 above, and the Ambrose-Kakutani Theorem. A measure-free version of the Birkhoff Ergodic Theorem for an automorphism of a standard space, attributed to Nadkarni and V. V. Srivasta, appears here for the first time: for a measurable set A, let D be the set of points from which the frequency of return to A in n steps does not converge as $n \to \infty$. Then D can be compressed by the automorphism (in a measure-free analogue of Hopf's definition alluded to above). Finally there are expositions of the aforementioned theorem of Mackey and Ramsey, the Glimm-Effros theorem giving conditions when a countable group of homeomorphisms of a Polish space admits a nonatomic invariant probability, and their combined implication that a countable group of automorphisms of a standard Borel space admits a nonatomic invariant probability if and only if there is no Borel cross-section of the orbits.

I found the book interesting and pleasant to read. It is written at the level of an advanced undergraduate or graduate text. There are nice little historical discussions interspersed.

Since Kakutani Equivalence and the Ambrose-Kakutani Theorem are both treated, it is surprising that there is no mention of the interpretation of Kakutani Equivalence in terms of time-change in flows, which is after all its main point; perhaps there could even be a Borel version.

One solecism: on page 71 the author uses the deep theorem of N. Friedman and D. Ornstein that a mixing Markoff shift is isomorphic to a Bernoulli shift to show that it is a K-shift. But the latter fact is actually much easier and was known long before the Ornstein-Friedman theorem.

One typo: on page 137, line -10 the word "of" should appear after "class".

Jack Feldman
University of California at Berkeley
E-mail address: feldman@math.berkeley.edu