BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 4, October 1996

Gröbner Bases: A computational approach to commutative algebra, by Thomas Becker and Volker Weispfenning (in co-operation with H. Kredel), Grad. Texts in Math., vol. 141, Springer-Verlag, Berlin and New York, 1993, xxii + 574 pp., \$49.00, ISBN 0-387-97971-9

An introduction to Gröbner Bases, by William W. Adams and Philippe Loustaunau, Grad. Stud. in Math., vol. 3, Amer. Math. Soc., Providence, RI, 1994, xiii + 289 pp., \$49.00, ISBN 0-8218-3804-0

Introduction

The theory of Gröbner Bases is a vivid example of how a simple idea used to solve one problem may become a key step in solving a great variety of other problems in different areas of mathematics and even outside mathematics.

When I teach the foundations of such a theory, I always like to compare the introduction of Gröbner Bases to the appearance of i as a solution to $X^2 + 1 = 0$. Once i is introduced and added to the reals, then the field of complex numbers arises. And all of a sudden a door is opened. Not only $X^2 + 1 = 0$ is solved, but also *every other polynomial equation* over the reals is solved.

Suppose, now, that we want to address the following problem: let S be a system of polynomial equations, namely,

$$f_1(X_1, \ldots, X_n) = 0, \ \ldots \ , f_r(X_1, \ldots, X_n) = 0$$

and suppose also that we have an additional polynomial equation $f(X_1, \ldots, X_n) = 0$. How can we decide if $f(X_1, \ldots, X_n) = 0$ is a *consequence* of the system S? Such a question can be translated in terms of ideal membership. Namely, we say that $f = 0$ is a consequence of S if f belongs to the ideal generated by $(f_1, \ldots, f_r)$. One of the reasons is that if I denotes such an ideal, then $f \in I$ implies that every solution for S is also a solution for $f = 0$.

The problem of deciding whether or not $f \in I$ is called the *Ideal Membership Problem*. It can be viewed as the search for a solution of $X^2 + 1 = 0$ in our analogy. And again, once the algorithmic tool is found to solve the Ideal Membership Problem, an incredible array of other problems can be addressed and *solved*. The key concept which does the trick is the notion of a *Gröbner Basis*.

As often happens, there are many who claim priority for this concept. In my view, the major step, which popularised the concept, was taken by Buchberger in the mid-sixties. Following a suggestion of his advisor Prof. W. Gröbner, Buchberger *introduced* the notion of a Gröbner Basis and, more importantly, he described an algorithm to *compute it*.

For many years the importance of Buchberger's work was not fully appreciated. Only in the eighties did researchers in mathematics and computer science start a deep investigation of the wide variety of applications that could be developed from the new theory. Many generalisations and applications to different problems were investigated. It immediately became clear that the theory of Gröbner Bases was about to become widely used and immensely popular in many areas of science.

1991 *Mathematics Subject Classification.* Primary 13P10.

What I have not yet said, but what is rather startling, is the utter simplicity of the fundamental ideas. *Simplicity and power*: two ingredients that make a perfect mix for the success of a theory.

For instance, researchers in commutative algebra and algebraic geometry benefited immediately from the appearance of specialised computer algebra systems such as Macaulay, CoCoA and Singular. The most fascinating feature of these systems was, and still is, their capability of tying together deep theories in mathematics as well as in computer science. Moreover, their key layout and fundamental engine is a good implementation of Buchberger's Algorithm for the computation of Gröbner Bases.

It was only in the nineties that the process of establishing computer algebra as an independent discipline started to take place. This fact contributed a great deal to the increased demand to learn about Gröbner Bases and inspired many authors to write introductory books on the subject. For instance, good sources are:

* Cox-Little-O'Shea: *Ideals, varieties and algorithms: An introduction to computational algebraic geometry and commutative algebra*, Springer-Verlag, 1992
* Mishra: *Algorithmic algebra*, Springer-Verlag, Berlin and New York, 1993
* Eisenbud: *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995

In this list we can certainly include the two books under review.

Gröbner Bases: A computational approach to commutative algebra by Thomas Becker and Volker Weispfenning (in co-operation with H. Kredel)

The structure of the book

As the authors make clear from the beginning, the book has a nonlinear structure, in the sense that it features three blocks, two of which embody six chapters with a strict linear dependence. The third block is a collection of four chapters, where the interconnections are less strict. The first chapters, ranging from 0 to 3, are meant to contain background material. But before delving into the subject, the authors declare their goal. They claim that, "*the development of new concepts and results in the area has now established* computer algebra *as an independent discipline that extends deeply into both mathematics and computer science*." This is a very important statement, which deserves to be discussed further in the scientific community, and I appreciate the fact that the authors are willing to address the issue. A little later in the introduction they say that, "*the purpose of the book is to give a self-contained, mathematically sound introduction to the theory of Gröbner Bases and to some of its applications, stressing both theoretical and computational aspects*." This choice leads to the decision "*to write a book that requires no prerequisites other than the mathematical maturity of an advanced undergraduate student*." The book clearly aims at becoming a standard textbook for a course on Gröbner Bases offered as part of a mathematics/computer science curriculum. Indeed, "*Algorithms are presented using a semi-formalism that is self-explanatory. ...*"

In Chapters 0 to 3 the basic background for a course in algebra is given. Natural numbers, integers, maps and the notion of a mathematical algorithm are the features of Chapter 0. In fact this sort of prelude already shows many of the positive and negative aspects of the book. The authors begin with the induction principle, but the presentation would not be sufficient for a reader who was not already familiar with the topic. This is quickly followed by the very concrete discussion of division of integers, followed by Cartesian products, maps, functions, injectivity, surjectivity, and cardinality. In all of this, propositions, exercises, comments, and examples are presented in a quite confusing style that blurs the concepts and makes the reading difficult. To me, the section devoted to the description of a mathematical algorithm is an unsettling compromise between precision and clarity; it is not necessary for an experienced reader and not enough for a beginner. But then Chapter 0 ends with the Notes! This is a common feature of all the chapters of the book, and what a beautiful feature it is! The style becomes terse, and the reader, beginner or expert, is pleased with the amount of relevant information about the historical background of the material presented in the chapter.

Chapters 1, 2, and 3 follow a similar pattern. In particular, Chapter 1 introduces basic notions such as groups, rings, ideals, and quotient rings.

Let's take a closer look at Chapter 2, which is devoted to polynomial rings. After having recalled the notion of polynomials in one indeterminate , the first statement in **boldface** is an exercise, which requires the reader to "*imagine a definition of polynomials in the variables* $X_1, \ldots, X_n$ *over* $\mathbb{Q}$." And then a great number of pages follow, where the reader tends to get lost between formal definitions and rigorous presentations of polynomial rings as monoid rings with all the universal properties. Then all of a sudden it is claimed that "*it is rather obvious that polynomials of* $R[X_1, \ldots, X_n]$ *can be represented on a computer...if and only if the same is true for the elements of* R." The reader gets confused. In any event a presentation like this is certainly not for beginners. Finally, unique factorisation is analysed as well as square-free decompositions and factorisation algorithms.

Chapter 3 deals with the notions of vector spaces and modules. The fundamental notion of syzygy is introduced here for the first time.

This ends the first block and the book enters its kernel. Chapter 4 is devoted to the notions of orders and abstract reduction relations. The expertise of the authors is evident, but they sacrifice clarity for an overly technical presentation of the ideas. For instance Section 4.3 deals with foundedness properties of relations. The reader is taken through dozens of subtle definitions, which tend more to be exhaustive than clear. An inexperienced reader might get easily lost, and an experienced reader would probably do better to skip almost this entire section.

Then we reach the very heart of the book, Chapter 5, which is devoted to Gröbner Bases. This chapter is, in my opinion, not adequate for anybody. The beginner will almost certainly get lost after a few pages. In my opinion, the decision to exclude syzygies from this stage of the presentation, instead of simplifying the matter, tends to hide its true mathematical nature. Therefore also the more advanced reader feels very uncomfortable. Having described in previous chapters all the necessary background, why should the authors spend more than fifty pages on the description of Gröbner Bases? Why should they spend many pages to describe the standard improvements of Buchberger's Algorithm, hiding the essence, which is *syzygies*?

Chapter 6 collects several applications of the notion of a Gröbner Basis. It is correctly proclaimed that Gröbner Bases are a cornerstone in commutative algebra, and the chapter aims at presenting a wide range of examples which support the claim. In particular it is shown how to use Gröbner Bases to compute syzygies, elimination ideals, intersections, colon ideals, and also how to compute in subrings of the polynomial ring. A special emphasis is given to zero-dimensional ideals. Very interesting is Section 6.4, which deals with the Word Problem; it is treated in the perspective of ring theory, and once more Gröbner Bases are shown to be the key for the decidability of the Word Problem in Abelian monoids and Abelian groups.

Chapter 7 deals with field extensions and the Hilbert nullstellensatz, which is presented with the aid of Gröbner Bases. In the section dealing with the height and the depth of prime ideals, it is surprising to see that the word *depth* is used in a totally non-standard way. What the authors call *depth(I)* is usually called the *Krull dimension of* R/I, and this is rather confusing. The chapter ends with two more fundamental applications, namely, to the implicitization of rational parametrizations and the invertibility of polynomial maps.

Many kinds of decompositions are considered in Chapter 8. More than fifty pages are needed to complete the task of treating the primary decomposition of zero-dimensional ideals. Then some generalisation to positive dimensional ideals is suggested, and finally the authors decide to switch to the computation of real zeroes. This is a totally different subject, and it is not clear why it should be included in the book. Chapter 9 deals with the computations that can be done in residue class rings, and again Gröbner Bases are shown to be the key. The final section claims to treat the notion of Hilbert function, which is certainly a basic object in commutative algebra and algebraic geometry; however, the authors completely overlook its real importance.

"Variations on Gröbner Bases" is the title of the final chapter. The topics are Gröbner Bases over PID's, homogeneous Gröbner Bases and homogenisation, Gröbner Bases for modules, systems of linear equations, standard bases and the tangent cone, and symmetric functions. Again I must say that, for instance, the sections devoted to the fundamental notions of homogeneous Gröbner Bases and homogenisation are written in a way that tends to hide the real essence of the subject.

Some comments

As I said before, the book is intended to be a self-contained computational approach to commutative algebra. And indeed the authors put a lot of effort in trying to be complete and elementary. They collected a great amount of material and decided to present it in a rather descriptive style.

But in my opinion the goals are not totally achieved. Many choices of topic are not adequate, and the style is not clear enough. Most of the time the exposition is confusing and the reader does not see the essence of the subject. The typographical style is also not very clear, although a great deal of care was taken to avoid typos. In conclusion, I would not recommend this book to students, but I would recommend it to experts, in particular for its beautiful historical notes and for its impressive bibliography.

An introduction to Gröbner Bases
by
William W. Adams and Philippe Loustaunau

The structure of the book

The first chapter, entitled "Basic Theory of Gröbner Bases", introduces the main questions addressed by the book and describes the theory of Gröbner Bases as a generalisation of both Gaussian elimination in the theory of linear systems and the division algorithm in the theory of univariate polynomial rings. The standard machinery (term orders, leading term ideals, the generalised division algorithm, rewriting rules) are described and used to define the notion of Gröbner Basis. S-polynomials and Buchberger's algorithm are then analysed.

It is worth noting that the notion of syzygy is postponed to Chapter 3. This choice has the advantage of keeping the level elementary and the drawback of hiding, for a while, the essence of Buchberger's algorithm. I am not totally convinced that this is a wise idea.

This chapter, as well as the subsequent ones, is enriched with many exercises. Among them there are some nice hints to more advanced topics like: the classification of term orderings; the theory of SAGBI bases; the equations of the projective closure of an affine scheme.

The second chapter, entitled "Applications of Gröbner Bases", treats the solution of some fundamental problems in commutative algebra. These include: the ideal membership problem, the radical membership problem, how to eliminate indeterminates, how to calculate the intersection of ideals, how to find the images of polynomial maps, and how to homogenise ideals.

The final sections present three applications to different areas of mathematics, namely: how to use Gröbner Bases to compute the minimal polynomial of an element in a field extension, how to decide if a graph is 3-colourable, and how to solve integer programming problems.

Here one should note that the choice of applications in such different areas gives a good idea about the versatility of Gröbner Bases.

The third chapter, entitled "Modules and Gröbner Bases", extends the theory from ideals to modules. The key notion introduced here is the notion of *syzygy*. The use of syzygies is highlighted, and it is shown that they can serve as an optimising device for Buchberger's algorithm and as an alternative tool for solving many computational problems already treated in the previous chapters.

Again many exercises help the reader become familiar with these deeper notions. The reader is instructed at the end of the chapter on how to compute $Hom(M, N)$, and there is a hint (not very clear in my opinion) on how to compute free resolutions.

The fourth and last chapter, entitled "Gröbner Bases over Rings", generalises the theory developed before to rings of polynomials with coefficients in a ring. It gives particular emphasis to the case where the coefficient ring is a PID (Principal Ideal Domain, like $\mathbb{Z}$).

Here there is an attempt to exit from the standard basics of the theory and show some more advanced topics. One should consider this chapter as optional; nevertheless it is well connected with the preceding sections. A great deal of work is done to convince the reader that passing from polynomials over a field to polynomials over a ring is not a "trivial" generalisation.

At the very end of the book there are two appendices. The first one deals with the systems used to make the computations presented in the book. The authors say that they used CoCoA more than other systems. Since I am the project manager of CoCoA, I was very pleased by that comment! Then they comment on the rules followed in the book to describe algorithms. The second one helps the reader to recall correctly the notions of well-ordering and induction. Was this appendix so necessary?

Some comments

The first claim of the authors is that the book has two goals:

"To give a leisurely and fairly comprehensive introduction to the definition and construction of Gröbner Bases;

"To discuss applications of Gröbner Bases by presenting computational methods to solve problems which involve rings of polynomials."

I would say that the goals are almost achieved. From the reader's perspective, though, I think the book would have been improved by a more lively presentation. For instance, the typographical setting does not help much to highlight the key points.

Later the authors say that *"computing is the very essence of the subject."* Indeed one of the major achievements of the theory of Gröbner Bases is that it provides a fundamental tool for widening the size and the depth of examples that can be computed. But of course the computation has to be carried out on a computer, and most theoretical mathematicians are not familiar with the concepts of computer science. The book neither attempts to cover this topic, nor does it point out that there is a great need for *easy* computational tools. Somehow the book says that computing is the essence, but it does not stress enough the need for a guide to computing.

Having made these criticisms, I must say that overall I consider this to be a good book, which I would recommend to well-motivated students as well as to researchers.

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