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Blow-up in quasilinear parabolic equations, by A.A. Samarski, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, vol. 19, de Gruyter Expositions in Mathematics, de Gruyter, Berlin and Hawthorne, NY, 1995, xxi + 533 pp., $198.95, ISBN 3-11-012754-7

Blow-up

A large part of the subject of partial differential equations concerns itself with singularities: their structure, their location in physical space, and their propagation. Frequently, the behavior of solutions away from these singularities can be guessed or is simply uninteresting.

Blow-up is a commonly occurring mechanism which generates singularities out of smooth initial conditions. It can already be found in the context of ordinary differential equations. For example, $X(t) = a/(1 - at)$ is a solution of the equation $x' = x^2$. This is the only solution corresponding to the initial condition $x(0) = a > 0$, and it does not exist for all t but only for t sufficiently close to the initial time. Furthermore, for ordinary differential equations, blow-up is equivalent to global nonexistence. The situation for infinite-dimensional initial value problems such as those arising from partial differential equations is more complicated. This in part is due to the coexistence of nonequivalent norms, each of which may serve as a measure for the size of a solution. For example, a space derivative in a certain direction may blow up while otherwise the solution might stay bounded. This is the case for the equations of gas dynamics where the solution may become discontinuous along a space-time surface, the so-called shock.

Most inquiries on blow-up have a common point of departure, regardless of origin. Whether the singularity stands for a hot spot (as in nonlinear heat equations), optical self-focusing or instabilities in plasma waves (as in nonlinear Schroedinger equations), an unstable solitary wave (as in the generalized KdV equation), a change of the homotopy class (as in the harmonic map flow), or a change of topological type (as in the mean curvature flow), the approach is the same: A differential inequality for a real-valued functional $F(u(t))$ of the solution u is derived. The inequality is then solved, subject to appropriate initial conditions at $t = t_0$, so as to obtain a lower bound for $F(u(t))$ that blows up at some finite time $t_1 > t_0$. If the definition of solution requires F to be finite for all time, then global nonexistence has been established. However, as Ball noted, it cannot in general be concluded that $F(u(t))$ itself blows up at some finite time, since the maximal half-open interval of existence of the solution may be $[t_0, t_{max})$, where $t_{max} < t_1$. If, on the other hand, an upper bound on $F(u(t))$ is derived that prohibits the functional to blow up in finite time, then one may be inclined to conclude global existence. Again the reasoning is not sound. Leray, in his 1933 *Acta Mathematica* paper on the Navier-Stokes equation, notes that the energy functional is a priori bounded, yet the solution may cease to exist because of blow-up of the second derivatives. In this way he was led to his notion of "solutions turbulente". To this day, mathematicians have not decided whether classical solutions for the Navier-Stokes equations may cease to be smooth.

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Leray's theory of turbulence is still a possibility. As Ruelle notes, strange attractors cannot account fully for all observable "singularities" in the physical flow.

Nonlinear heat equations

Research on blow-up of solutions in nonlinear heat equations, the subject of the book under review, is partly the outcome of efforts to understand the Navier-Stokes problem. In fact it was Fujita, an authority on that subject, who initiated the study of blow-up for

$$u_t = \Delta u + f(u) \tag{1}$$

where $f: \mathbb{R} \to \mathbb{R}$ is smooth, positive, increasing and strictly convex. Another very influential paper on the subject of nonlinear partial differential equations has been Gelfand's. This paper contains a section on combustion involving the equation

$$u_t = \Delta u + e^u. \tag{2}$$

Since the diffusion operator uniformizes, it may seem paradoxical that these equations possess solutions that blow up. After some thought, however, one realizes the competition between the Laplacian and the nonlinear term, which favors localization.

The equations (1), (2) were investigated in the 70's and 80's, during which a lot of progress was made on blow-up to answer the three fundamental questions: <u>When</u>, <u>Where</u> and <u>How</u>. The best results were obtained under Dirichlet conditions for the ball or the whole space. The monograph of Bebernes and Eberly gives a very good account of these problems, especially on equation (2). The tool employed in these investigations, besides differential inequalities, is the comparison/maximum principle. Its Sturmian refinement allows for very detailed results in one space dimension.

During this same period, singular and degenerate heat equations were also intensively investigated. A major factor in this investigation was the Stefan Problem, describing the melt of ice in a bath of water, which can be cast in the form

$$u_t = \Delta\beta(u) \tag{3}$$

where $\beta(u)$ is an affine monotone graph with a vertical part. Another motivating factor became the phenomenon of diffusion in porous media which is described by

$$u_t = \Delta(|u|^{m-1}u), \quad m > 1. \tag{4}$$

By carrying out the differentiation in (4), it can be seen that the principal part of the operator drops out at the interface $u = 0$. This in turn suggests that localized disturbances propagate with finite speed. The explicit solution of Barrenblatt and Pattle confirms this. The finite speed of propagation is the single most important fact about (4) and makes this equation a very interesting object of study.

The book under review

The book under review addresses blow-up for quasilinear heat equations, emphasizing the porous medium operator for $m > 0$ and including a variety of source terms. The semilinear case (1) is also treated in detail as are some related systems. There is also material on the asymptotic behavior of

$$u_t = \Delta u - u^\beta, \quad \text{on } \mathbb{R}^n, \tag{5}$$

whose solutions do not blow up. For most equations, the domain is either $\mathbb{R}^n$ or the half line.

This book is a monograph describing work on qualitative aspects, carried out during the 70's and 80's, by a prominent group of researchers in Russia. The central theme of the book deals with the extent of the explosive event (and also with the "where" and the "how"). It is introduced in Chapter 3 in terms of the following example.

$$\begin{cases} u_t = (u^\sigma)_{xx}\,, & x > 0,\ \sigma = \text{const.}\ > 0 \\ u(0,t) = (T-t)^{-\frac{1}{\sigma}}, & t < T \end{cases} \tag{6}$$

This initial-boundary value problem has a separable solution given by

$$u_s(x,t) = \begin{cases} (T-t)^{-\frac{1}{\sigma}}(1 - x/x_0)^{\frac{2}{\sigma}}, & 0 < x \le x_0 \\ 0, & x > x_0 \end{cases}$$

where

$$x_0 = [2(\sigma+2)/\sigma]^{\frac{1}{2}}\,.$$

The main features of the solution are

a) for $0 \le x < x_0$ the temperature $u_s(x,t)$ goes to infinity as $t \to T^-$;
b) $u_s(x,t) \equiv 0$ for all $t \in (0,T)$ for any $x \ge x_0$.

This example shows that the process of heat transfer is localized in the finite domain $0 < x < x_0$, even though in that domain the temperature grows without bound as $t \to T$.

The line of inquiry goes as follows: First, explicit solutions are constructed, mostly self-similar. Then, by comparison methods, it is established that these explicit solutions accurately represent the asymptotic behavior of fairly general solutions.

Known comparison techniques have been systematized and further extended by the introduction of novel ideas, comparing solutions which belong to different equations. Chapter 5 deals with such a comparison theorem and its applications. To give the flavor of these results, we now describe a corollary. First, we introduce *critical solutions*. These are special solutions defined by the condition $u_t > 0$. They are well-known objects, easily constructed, and not hard to analyze since they do not oscillate in time.

Corollary. *Consider the pair of equations*

$$u_t^{(\nu)} = \operatorname{div}\left(k^{(\nu)}(u^{(\nu)})\nabla u^{(\nu)}\right) + Q^{(\nu)}(u^{(\nu)}), \quad \nu = 1,2. \tag{7}$$

Assume that $u^{(2)} \ge u^{(1)}$ on the parabolic boundary, that $u^{(2)}$ is critical, and that the following relations hold:

$$k^{(2)} \ge k^{(1)}, \quad \left(k^{(2)}/k^{(1)}\right)' \ge 0, \quad Q^{(2)}k^{(1)} - Q^{(1)}k^{(2)} \ge 0.$$

Then

$$u^{(2)} \ge u^{(1)} \text{ on the parabolic cylinder.}$$

Later in Chapter 6 *approximate self-similar solutions* are employed. Their main feature is that they do not satisfy the equation, yet they correctly describe the asymptotic behavior of the problem under consideration. The idea goes as follows:

The elliptic operator A in the evolution equation (which by assumption does not have an appropriate self-similar solution) is decomposed into a sum of two operators,

$$A(u) = B(u,t) + [A(u) - B(u,t)] \tag{8}$$

so that the equation

$$u_t = B(u,t) \tag{9}$$

admits a self-similar solution. The important fact is the smallness of the perturbation $A - B$ along this particular solution.

Example.

$$\begin{cases} u_t = A(u) \equiv (k(u)u_x)_x, & x > 0 \\ u(0,t) = u_1(t) \to \infty & \text{as } t \to T. \end{cases} \tag{10}$$

It is shown that for a wide class of k's and u_1's, a correct choice for (9) is provided by the Hamilton-Jacobi equation

$$u_t = \frac{k(u)}{u+1}[u_x]^2. \tag{11}$$

Notice that (11) is not easily guessed.

Other comparison techniques include Sturmian theorems for counting intersections of solutions, possibly of different equations. In this way, the behavior of general solutions is described in terms of special families, depending on a few parameters. Also, other methods of reduction to a finite number of parameters are considered, and therefore this book could very well be viewed as a study of the underlying finite-dimensional dynamics and their stability, with the emphasis mostly on the analytical aspects.

Overall this is a well-written, well-organized book. It contains a sufficient amount of basics, and so it should be accessible to graduate students. One of its strong points is the tremendous wealth of explicit examples. It should prove invaluable to researchers of blow-up in the context of parabolic problems, and together with the book of Bebernes and Eberly it covers a large part of what is known in the subject.

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