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The adjunction theory of complex projective varieties, by Mauro C. Beltrametti and Andrew J. Sommese, de Gruyter Expositions in Math., vol. 16, de Gruyter, Berlin and Hawthorne, NY, 1995, xxi + 398 pp., $89.95, ISBN 3-11-014355-0

Together with Fujita's *Classification theories of polarized varieties* (London Mathematical Society Lecture Notes series **155**), this book will be the bible for researchers in the field as well as for those outside of the field, like me, who would like to know what the theory of polarized varieties is or what has been done in the subject, which the authors alternatively call "the adjunction theory".

Let us quickly go over the contents of the book.

In Chapters 1–4, the book tries to provide a concise and quick introduction to the background material as an appetizer for the main course to come. It contains several interesting results which one may not find in a regular textbook of algebraic geometry. But the reader of the book is advised to taste it with a grain of salt. There are some inaccuracies to warn you not to swallow the whole thing as presented. In Chapter 1, for example, I can list the following:

(i) The symbol "$\approx$", which is defined to represent the $\mathbb{Q}$-linear equivalence at the beginning, is later often misused to represent the linear equivalence.

(ii) The definition of "ϕ-nef" is given as an analogy to the one for "ϕ-ample". But this differs from the right definition in a subtle but definitive way: One finds a line bundle whose intersection with any curve contracted by ϕ is always nonnegative and hence "ϕ-nef" in the right definition but which can never be absolutely nef no matter how much one tensors the pullback of an ample divisor from downstairs by, for example, looking at $\phi : E \times E \to E$ for a general elliptic curve.

(iii) The proof of Lemma 1.1.4 is wrong without the assumption of singularity being Gorenstein. The claimed isomorphism

$$0 \to p_*(mK_{\bar{V}}) \to mK_V$$

can be given a counterexample by looking at some surface quotient singularity.

(iv) The proofs are sometimes roundabout and superflous: in the proof of Lemma 1.1.3 the book proves local triviality of some line bundle to verify that a certain morphism factors through another morphism. But this is irrelevant, and the factorization of the concern follows directly. In the proof of Lemma 1.6.7 after showing

$$\dim H^2(Z_V, \mathbb{Q}) = 2 > \dim H^2(X, \mathbb{Q}) = 1,$$

the required intersection number $L^2 = 1$ follows immediately as the difference of these two without Euler characteristic computation of the book.

(v) In the statement of Kawamata Rationality Theorem in the book

$$\text{“}r\tau = u/v\text{”}$$

should be

$$\text{“}r/\tau = u/v\text{”}.$$

(vi) In example (1.8.2.2) the book gives a nef line bundle L with $L^{n-k}H^k > 0$ for an ample divisor H and $k > 0$, but no multiple of L has any sections. But L is

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not k-big in the sense defined on page 5 for any $k > 0$, in contradiction to what is claimed at the beginning of the section.

These mistakes might be taken as a form of entertainment for readers to confirm the statements for themselves in order to advance to the main course rather than as an obstacle.

Chapter 2 recalls various vanishing theorems derived from the positivity (ampleness) conditions, making several generalizations in the presence of singularities and/or replacing the notion of "ampleness" with "k-ampleness". While these generalized statements are technically useful for the purpose of the later chapters, I recommend that readers try to come up with a proof for themselves as an exercise instead of reading routine ones in the book. In case the reader fails to do so, the book provides a good answer with detailed inductions.

Chapter 3 may be best enjoyed with Fujita's book and some of his papers at hand.

Chapter 4 provides a quick ride to the theory of extremal rays of Mori and base point freeness of Kawamata. For those who have never heard of the Minimal Model Program or its related results, this chapter may be a list of unmotivated theorems. But the conciseness of the summary here may help the reader to swallow it as an appetizer.

Though my comments on Chapters 1–4 may lean toward the critical side, these myopic criticisms should be ignored, especially when one thinks of the overwhelming convenience and usefulness of this brief coverage of the prerequisites.

The main course of the book starts with Chapter 5. Throughout the course, one of the major strategies is to classify the polarized varieties $(X, \mathcal{L})$ according to the behavior of adjoint bundle $K_X + t\mathcal{L}$ with t passing through critical nef value $\tau(\mathcal{L})$. This culminates in the study of what the authors call "first reduction" and "second reduction" in general adjunction theory in Chapter 7. Every time t hits integer values $t = \dim X, \dim X - 1, \dim X - 2, \ldots$, it sorts out the polarized varieties into three categories: 1° general ones with $K_X + t\mathcal{L}$ ample, 2° several special varieties with $\tau = t$ where we expect to know their rigid structures, or 3° the ones whose nef value birational morphisms $\phi_{K_X + t\mathcal{L}} : (X, \mathcal{L}) \to (Y, \mathcal{L}_Y)$ make the reduction to the variety Y with $K_Y + t\mathcal{L}_Y$ ample. Thus, aside from the exceptional few special varieties, we proceed with our analysis with $t-1$ inductively. When $t = \dim X - 1$ or $\dim X - 2$, the process is called "first reduction" or "second reduction" respectively. The bigger t is, the more rigid structures we expect to be able to describe on those special varieties, and it is for this description that the results of Chapters 5 and 6 are utilized. A more detailed study is given in Chapter 12 with the restriction $\dim X = 3$.

Chapters 8, 10, and 11 are written under the theme of "Classical Adjunction Theory", and starting from some classical results they collectively present the research accomplishments due mainly to the second author and provide a good reference for the current stage of the field inspired by his results and those of his coworkers. Chapters 13 and 14 focus on varieties with some special features. The proofs are usually presented in detail and at a leisurely pace to provide good reading material. Various results quoted without proofs will make the book more valuable than uncomfortable for the purpose of references and future studies.

Here is my prejudice: Though the book left me with an enormous amount of information about WHAT the adjunction theory is all about, it provided very little to answer my original question, WHY adjunction theory?

Fujita's comment in the introduction of the book—"In my opinion, God did not make abstract varieties but polarized varieties"—explains to a great extent the motivation and philosophy behind the theory: study the intrinsic geometry of a variety X through the God-given polarization L (an ample divisor) and the canonical divisor K_X. His beautifully written book appeals strongly to the mind of the reader with this theme. This gives justification to the study of the combinations $tK_X + sL$ and critical values (e.g., nefvalue and sectional genus, etc.). But then I hear some incredulous voice whispering, "How about the other divisors arising from the various operations of these divisors?" For example, take a member $D \in |mK_X|$, and add its reduced part to the canonical divisor to obtain $K_X + D_{red}$. Isn't this divisor as much God-given as L, K_X, or any other combination $tK_X + sL$? The Iitaka category of logarithmic pairs (X, D) seems to provide a more natural working ground than making the restrictive condition of D be ample. Eventually the divisor $K_X + D_{red}$ plays a crucial role in Kawamata–Miyaoka's proof of Reid's Abundance Conjecture in the framework of Mori's program combined with Iitaka's, and it seems that it is within this framework that the true importance of the adjunction theory is also being carved out. Accordingly the book spends a substantial number of pages quoting theorems from Mori's program, citing notions such as "terminal singularities". But these notions have little persuasive power in the book, where they are borrowed for the sake of generalization rather than for mathematical necessity from within.

There is no doubt that the adjunction theory or the classification methods of polarized varieties will occupy one of the central positions in the theory of algebraic varieties. It will surely not be replaced by Mori's program, and its importance will be more and more emphasized in its fundamental role. But the book left me puzzled about the direction in which the future study of the subject is going or the inner motivation which urges one into the subject. It is up to the reader of the book to decide whether I should perish from my prejudice or not, and the book is well written for the reader to study and accumulate a basis for the judgment.

Kenji Matsuki
Purdue University
E-mail address: kmatsuki@math.purdue.edu