BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 4, October 1996

Topological theory of dynamical systems, by N. Aoki and K. Hiraide, North-Holland Math Library, vol. 52, North-Holland, Amsterdam, 1994, viii + 416 pp., $128.50, ISBN 0-444-89917-0

The general field of dynamical systems goes back (at least) to Poincaré, but many different facets have developed in recent years both within and outside of mathematics. One of the important directions of development is the theory of hyperbolic differentiable dynamical systems which is based on the work of Smale (see Smale [9]). The book by Aoki and Hiraide is a topological development of some of the main ideas of hyperbolic differentiable dynamical systems and might more aptly be titled *Topological theory of* hyperbolic *dynamical systems*.

Akin's book [1] is another recent book about the part of topological dynamics which is motivated by differentiable dynamical systems. By contrast, Aoki and Hiraide give a coherent collection of theorems on the more focused subject of topological hyperbolic dynamics. Both of these books deal with noninvertible maps in addition to homeomorphisms. In this regard, Akin takes it a step farther and treats relations which are not functions, while Aoki and Hiraide stick with noninvertible maps which are more easily understood by most mathematicians. The review of Akin's book by J. Auslander [3] should be consulted for a brief history of topological dynamics and general references to this subject. There have been many texts on this theory of differentiable dynamical systems, including the two more recent textbooks by Katok and Hasselblatt [6] and Robinson [7].

Turning to a more detailed discussion of hyperbolic dynamics, the work of Smale grew out of the desire to understand which systems do not change their basic dynamics under perturbation: structurally stable systems. One early example of such a stable system was the horseshoe which was a model for the complicated dynamics found by Cartwright and Littlewood in a forced nonlinear oscillator. A second example was a linear hyperbolic toral automorphism (induced by an integer matrix for which no eigenvalue has absolute value equal to one) originally suggested by R. Thom. Anosov studied the whole class of flows and diffeomorphisms which bear his name and are hyperbolic on the whole manifold: this class includes the hyperbolic toral automorphisms as a special case [2]. Smale abstracted these and other examples into the assumption of hyperbolicity of an invariant set. A set Λ is called *invariant* for a diffeomorphism f provided $f(x) \in \Lambda$ for $x \in \Lambda$. The invariant set is said to have a *hyperbolic structure* provided at each point $x \in \Lambda$ the tangent space splits into two subspaces, one of which is expanded by the derivative of iterates of f and the other is contracted. An invariant set which is hyperbolic is then preserved by small perturbations, i.e. the system is structurally stable. Anosov [2], Sinai [8], and Bowen [4] showed that a diffeomorphism with a hyperbolic invariant set has another feature called the pseudo orbit tracing property. (Unfortunately there are two sets of terminology: pseudo orbits are the same as ϵ-chains, and tracing is the same as shadowing.) An *ϵ-pseudo orbit* (or *ϵ-chain*) is a sequence of points $\{x_i\}_{i=-\infty}^{\infty}$ such that $d(x_{i+1}, f(x_i)) < \epsilon$: it almost makes up a true orbit. Such an ϵ-pseudo orbit is said to be *η-traced* (or *η-shadowed*) provided there is

1991 *Mathematics Subject Classification.* Primary 58F10, 58F15.

a point y such that $d(x_i, f^i(y)) < \eta$ for all i. For a hyperbolic invariant set, given any $\eta > 0$ there is an $\epsilon > 0$ such that any ϵ-pseudo orbit in Λ can be η-traced by a point $y \in \Lambda$. This property is called the *pseudo orbit tracing property*, POTP (or the *shadowing property*). One way the book under review is more topological in its perspective is that many of the theorems assume that a continuous map or homeomorphism has the POTP rather than assuming it is differentiable with a hyperbolic structure.

Another key topological assumption used in Aoki and Hiraide's book is expansivity: a set Λ is said to be *expansive* provided there is an $r > 0$ such that if $x, y \in \Lambda$ with $x \neq y$, then there is an n such that $d(f^n(x), f^n(y)) \geq r$. This property is closely related to the property of sensitive dependence on initial conditions which is often referred to in connection with chaos. See Devaney [5] or Robinson [7].

A homeomorphism $f : T^n \to T^n$ is called a *topological Anosov homeomorphism* provided it is expansive and has the POTP. Walters' Theorem is one among many that this book gives about topological Anosov homeomorphisms: a topological Anosov homeomorphism f is topologically stable; i.e., for any homeomorphism g which is C^0 near f, g is semi-conjugate to f. They also prove that any topological Anosov homeomorphism is topologically conjugate to a hyperbolic toral automorphism (induced by a matrix with integer entries).

The authors carry over many of the theorems to noninvertible maps. The *inverse limit space* M_f is the set of all orbits $(x_i)_{i=-\infty}^{\infty}$ where $f(x_i) = x_{i+1}$. They call a noninvertible map f *c-expansive* provided the map induced by f on M_f is expansive. A continuous surjection is called a *topological Anosov map* provided it is c-expansive and has the POTP. They prove classification theorems for topological Anosov maps which are analogous to the above theorem for topological Anosov homeomorphisms. The fact that this book contains many of these theorems for noninvertible maps is the aspect which I personally find most useful. Many papers are written which try to carry over some theorem in hyperbolic dynamics to noninvertible maps. Any future author of such a paper would be advised to consult this book.

I find the book develops the ideas in ways that are not available in other textbooks or monographs. There are many results in the book that I did not know (even if many of them seemed "reasonable"). I also found the book readable, and I think it makes a very useful reference work for a researcher in hyperbolic dynamical systems. The book presents all the ideas necessary to understand the results, so it would be possible for a graduate student or someone else to learn the theory from this one source. However, I think most people would benefit from reading a textbook on differentiable dynamical systems at the same time to gain insights from further examples and from connections with differentiable assumptions.

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