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Linear and quasilinear parabolic problems, Vol. I: Abstract linear theory, by Herbert Amann, Monographs Math., Birkhäuser, Basel, Boston, and Berlin, 1995, xxxv+335 pp., $119.00, ISBN 3-7643-5114-4

Some years ago I was invited by Herbert Amann to give a lecture at the Seminar on Nonlinear Analysis that he led (together with the late Peter Hess) at the Institute of Mathematics of the University of Zürich. As the lecture that I was going to give dealt with *linear* problems, I pretended to apologize, saying that I was not outside the subject of the seminar, since linear analysis is a particular case of nonlinear analysis. Obviously this was only a joke, because the scope and methods of linear and nonlinear analysis are too different, but there was a kernel of truth: one cannot study nonlinear problems without a deep knowledge of the linear case. This book is demonstration of that.

Parabolic equations and systems are used as mathematical models of a great variety of irreversible phenomena which are studied in the "natural sciences" (i.e. physics, chemistry, biology). Their formal aspect is

$$\partial_t u = \mathcal{A}u + f, \tag{1}$$

where u is the unknown function, f is given, the real variable t can be interpreted as time, and $\mathcal{A}$ is a differential operator in another set of real variables, the so-called "space variables" $x_1, \dots, x_N$. Equation (1) is *linear* if the datum f and the coefficients of the differential operator $\mathcal{A}$ depend only on the independent variables $t, x_1, \dots, x_N$; it is termed *semilinear* if f depends also on u, *quasilinear* if the coefficients of $\mathcal{A}$ also are allowed to depend on u.

This (very rough) description actually applies to any type of *evolution* equation, that is, to any equation where a (not necessarily first-order) derivative in time is faced by a differential operator in the space variables; the condition for parabolicity of equation (1) is that the operator $\mathcal{A}$ be *elliptic*. Without entering into detail, the latter is a condition involving the positivity of the real part of some homogeneous polynomial associated with the principal part of $\mathcal{A}$.

There is an enormous amount of literature on (linear, semilinear and quasilinear) parabolic problems, the greatest part of which is concerned with the case of a single second-order equation (that is, second order in the derivatives with respect to the space variables). In this case, the classical approach to the initial-boundary value problems for equation (1) essentially makes use of: (i) maximum principle to obtain uniqueness results, (ii) potential theory for existence results, and (iii) a lot of hard inequalities for regularity of solutions. A sytematic exposition of this approach can be found in the book [5]. This approach, however, does not exploit thoroughly the special nature of the time variable: t is a variable like the others, except that it has a different "weight".

Another attitude towards equation (1) (an attitude, by the way, which is responsible for at least part of the interest in functional analysis and operator theory in the last fifty years) is that of looking at it as an *ordinary* differential equation, where u and f are functions of t alone, with values in some Banach space of functions of

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the space variables. The choice of this space is obviously of paramount importance; the (linear) differential operator $\mathcal{A}$ then becomes an unbounded linear operator A acting in that space, and boundary conditions (if any) that accompany equation (1) are included in the definition of the domain of A. When the differential operator is elliptic and the space is chosen judiciously, it happens that with a remarkable freedom in the setting of the boundary conditions, A will be the infinitesimal generator of an analytic semigroup.

Semigroup theory is suited to the study of the linear autonomous Cauchy problem

$$\begin{cases} u'(t) = Au(t) \quad (t \geq 0) \\ u(0) = u_0 \end{cases} \tag{2}$$

(where *autonomous* means that A is independent of t): the celebrated Hille-Yosida theorem gives necessary and sufficient conditions, expressed in terms of spectral properties of A, for the existence of a family of operators $t \mapsto \exp(tA)$ $(t \geq 0)$ such that $\dfrac{d}{dt}\exp(tA)u_0 = A\exp(tA)u_0$, at least when u_0 belongs to the domain of A. When this semigroup is *analytic* (i.e. can be extended to a holomorphic function on some sector of the complex plane containing $\mathbf{R}^+$), the resolvent set of A must contain a sector strictly larger than a half-plane, which witnesses the irreversible character of the phenomena described by equation (2).

In 1953 T. Kato published the first paper concerning the nonautonomous equation

$$u'(t) = A(t)u(t) + f(t). \tag{3}$$

The aim was to build an *evolution operator*, that is, an operator-valued function $(t, s) \mapsto U(t, s)$ such that $U(t, t) = I$ (the identity operator) and $\dfrac{\partial}{\partial t}U(t, s)u_0 = A(t)U(t, s)u_0$, at least when u_0 is regular enough. In the last forty years that was done by many authors, under weaker and weaker assumptions on the dependence of $A(t)$ on t; at least in the *parabolic* case (i.e. when the operators $A(t)$ generate analytic semigroups) the usual condition is some kind of Hölder continuity of the resolvent operators $(\lambda + A(\cdot))^{-1}$. A different approach was proposed in the late 1960's by P. Grisvard for the parabolic case (while G. Da Prato carried on the so-called hyperbolic case: see their joint recapitulatory paper [1]): the matter is to rise another step in the scale of abstraction and write equation (3) in the form

$$(\tilde{A} + \tilde{B})u = f \tag{4}$$

where, if E is the Banach space where $u(t)$ and $f(t)$ live, $\tilde{A}$ and $\tilde{B}$ are linear operators acting in some Banach space X of E-valued functions: one of them, say $\tilde{B}$, is the operator of derivative with respect to time, with an appropriate domain, and the other is the superposition operator $u \mapsto A(\cdot)u(\cdot)$.

A theory so elaborate, with contributions from so many authors, periodically needs an organic exposition. This has been done several times in the last thirty years; and besides the book under review, omitting the books more particularly devoted to semigroup theory, one should mention at least (in chronological order) [4], [3, Part II], [7], [2] and [6]. However, with the exception of the last-quoted book and of the older text by Friedman, the other books are not specifically concerned with abstract parabolic equations but more generally with evolution equations in Banach spaces, including those in which the operators $A(t)$ generate nonanalytic semigroups. This is not the only difference between Amann's book and the others (including Lunardi's). First, Amann gives as known the theory of analytic semigroups and treats the autonomous equation only in Chapter III (very long, however), where he expounds an abstract theory of maximal regularity in different function spaces for the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = u_0, \end{cases}$$

reserving the major part of the text to the study of the nonautonomous equation (existence, uniqueness and regularity of solutions, construction of the evolution operator); second, and above all, this is the first volume of a treatise which is expected to have three volumes and whose final aim is a systematic exposition of results concerning *quasilinear* parabolic problems, i.e. problems to which the author has given fundamental contributions in the last several years.

This feature explains some choices of the author and the general style of the book. Besides the omission of analytic semigroup theory and the (wise) decision of listing without proofs the needed results from interpolation theory, one can remark that Amann neglects some results in the case where the domains of the operators $A(t)$ may vary with t, because the assumptions necessary to obtain such results are too heavy to be useful in the applications to nonlinear problems. As for the style of the book, this is something less touchable, but everywhere there is constant care by the author to give uniform estimates for families of operators (or of operator-valued functions), quantitative results (in perturbation theorems, for instance), and estimates of the constants appearing in formulas. In short, one understands that the author is loading and pointing his (linear) guns to fire at his (nonlinear) targets.

This volume contains the first five chapters of the treatise (references to subsequent chapters are scattered here and there). In the first chapter the author collects some results on generators of analytic semigroups and gives a summary (without proofs) of that part of interpolation theory that proved to be useful in the study of parabolic problems. The second and fourth chapters are devoted to the study of the Cauchy problem for the (parabolic) equation (3) in the cases of constant domains and variable domains respectively (but in the latter case assuming that some interpolation space between the domain of $A(t)$ and the surrounding Banach space is independent of t). In Chapter III maximal regularity is studied: i.e. (roughly speaking) spaces of Banach-valued functions and operators A are looked for such that the solutions of the equation $u' = Au + f$ enjoy the property that u' and Au have the same regularity as f. In Chapter V the evolution equation is studied in scales of interpolation-extrapolation Banach spaces: this provides an abstract framework to the study of weak solutions and has a technical utility in the study of regularity problems.

In my opinion, reading this book might not always be easy (to a nonspecialist of the subject, for example) because of the conciseness of notation, very consistent and precise indeed, but sometimes a little cruel. However, this is a very small flaw. Apart from that, this is an excellent reference book, very well conceived and written. It witnesses the author's thorough knowledge of the literature and remarkable coherence and ability in the presentation of the subject. The proofs are detailed and clear, and the logical thread of chapters and sections is explained in concise remarks. A very long and very interesting introduction discloses the author's intentions for the next volumes of the treatise and explains the reasons for many technical choices and devices (why the author prefers to look for spacial regularity in Sobolev spaces rather than in spaces of continuous functions, where interpolation theory appears naturally in the study of regularity, why the problems with constant domain are in some sense more important for the study of quasilinear problems than problems with variable domains, why one cannot get rid of the problems with variable domains, when results of maximal regularity for the linear equation become necessary, etc.).

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Alberto Venni
Universita degli Studi di Bologna
E-mail address: venni@dm.unibo.it