BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 4, October 1996

Introduction to the modern theory of dynamical systems, by Anatole Katok and Boris Hasselblatt, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge Univ. Press, 1995, xviii+802 pp., $79.95, ISBN 0-521-34187-6

The dynamical systems referred to in the title are mathematical models for systems with time-dependent states and whose successive states are determined by a deterministic and time-independent rule. As examples one may think of the planetary system, electrical circuits, mechanical devices, and also of many particle systems, like gases in the sense of classical (non-quantum) mechanics. One assumes that the state spaces of these systems are finite-dimensional (states may be specified by, e.g., the positions and velocities of finitely many particles). The time evolution is then given by either a differential equation (in the case of continuous time) or by a map (in the case of discrete time). Discrete time may look artificial, but though most theoretical problems and concepts are essentially the same for continuous and discrete time, the case of discrete time is often easier to formulate. This makes systems with discrete time attractive from the mathematical viewpoint. Situations where processes with discrete time naturally occur are, e.g., iterative processes in numerical computations (the nth state being the result after n iterations) and models describing the dynamics of populations (the nth state being the size of the nth generation).

The authors, and especially the first author, made important contributions in the area of dynamical systems, especially in the ergodic theory of differentiable dynamical systems and the theory of (nonuniform) hyperbolic systems. Due to their background they are well informed about both the Russian and the Western literature. This enables them to treat many subjects in such a way that even specialists will find original aspects and new points of view.

The authors give a self-contained and very carefully organised introduction to extensive parts of the recent developments in dynamical systems. The level and scope of the book can be best described, as the authors do in the preface, by noting that it can be used as a text for various types of graduate courses in the following way: the first four chapters contain material which is essential for all that follows; by an appropriate selection of the following chapters one obtains coherent graduate courses on, e.g., variational methods in classical mechanics, hyperbolic dynamics, twist maps and applications, and ergodic theory of smooth dynamical systems. Each of these subjects is developed in considerable depth.

Still, the authors did not cover, and did not want to cover, all of what belongs to dynamical systems (according to many). A notable example of this is the fact that famous examples like the Lorenz and the Hénon attractor are not even referred to in the bibliography and only discussed in a note giving some background information on the Smale attractor. Other examples of such "numerically defined attractors", like the ones found by, e.g., Rössler, Shilnilov and Zaslavski are not mentioned at all. Another subject which is hardly covered is complex dynamics (complex in the sense of complex numbers), associated with objects like the Mandelbrot set.

1991 *Mathematics Subject Classification.* Primary 58Fxx, 34Cxx, 53C22.

This choice is probably because the authors regard the theory of dynamical systems as a purely mathematical discipline, independent of the motivations and inspirations it received from fields outside mathematics and independent of developments it stimulated in other areas of mathematics. Let me indicate some of the instances where interactions with developments outside mathematics were of great importance for the development of the theory of dynamical systems.

One of the first substantial contributions to the modern theory of dynamical systems was Poincaré's prize-winning essay of 1890. The fundamental question in that paper is the stability of the solar system. Of course, by that time it was known how to calculate the future positions of the planets on a time scale of several years, and even to do so numerically with great precision, but here the question was about the entire future of the solar system: is it possible, according to the laws of mechanics, that the (small) interactions between the planets produce in the long run big effects (e.g., causing a planet to escape from the solar system), or will the effects of these interactions in average cancel approximately, even after arbitrarily long time intervals? It is not obvious how to formulate this question in a mathematically rigorous way, but that is not our concern here. The main point is that we have a question which cannot be (positively) solved by any numerical approximation of a solution for a bounded time interval. The properties of solutions of differential equations, or of orbits of dynamical systems, which depend on their limiting behaviour as time goes to infinity are called their asymptotic properties. These properties are often in terms of recurrence, i.e., in terms of the limit points of the orbits in question.

The main outcome of the investigations of Poincaré was that, to his own surprise, he found that the laws of mechanics (and he saw no reason that even the motion of three bodies under the influence of gravity would be much simpler) lead in general to recurrence of an unexpectedly complicated nature—that is what we nowadays call "chaos".

In the first half of this century, through the investigations of Van der Pol, Cartwright and Littlewood, and others, it became clear that these complications were not restricted to the somewhat idealized theory of frictionless mechanics, which is the basis of the description of the solar system used by Poincaré, but also appear in more earthy situations, like electronic circuits and mechanical devices with friction.

Although it was clear that one had to take into account the possibility of extremely complex behaviour, the behaviour itself was very poorly understood, and it was not clear whether these complications would only appear for exceptional solutions, say, corresponding to a set of initial states of total measure zero or that these complications could occur in a persistent way.

This situation was changed drastically in the mid '60s by two independent developments. One was Smale's investigation of the geometric theory of differentiable dynamical systems which gave geometrical insight in, at least examples of, systems with complicated asymptotic behaviour. The systems which can be understood through Smale's approach are called hyperbolic systems. This analysis implied that for certain classes of dynamical systems this complicated behaviour is not exceptional (though it is not clear whether these classes of dynamical systems included the ones of practical interest).

Another was Lorenz's discovery, when simulating a very simplified model of thermal convection (which in its turn is a simplification of the dynamics in the atmosphere in which Lorenz, as a meteorologist, was primarily interested), that the behaviour of the numerical solutions, in his model, are seemingly random and even numerically irreproducible (except when using exactly the same rounding-off and discretization procedures). This is now understood as a manifestation of the same type of asymptotic complexity (chaos) which was present in Smale's geometric examples and in the planetary system studied by Poincaré.

This example of Lorenz provided a strong indication that the complicated asymptotic behaviour which was known theoretically since Poincaré and geometrically understood, at least in certain examples, since Smale really occurs in systems of practical interest. However, the mathematical relevance of the numerical experiments went further: it turned out that all the numerical examples (at least the early ones) which showed the complex asymptotic behaviour referred to above did *not* satisfy the hypotheses, necessary for the geometric theory of Smale to be valid. So they provided strong evidence for the necessity to extend the theory. Such extensions form an important subject in the research on dynamical systems of these days.

Another development, also starting around the beginning of the century, is related to the ergodicity postulate in statistical mechanics. In this theory one deals with systems (like gases or fluids) consisting of an extremely large number of particles (molecules) for which it is much too complicated, and even not useful, to investigate particular solutions. Instead, one is interested in statements about the statistical, or averaged, properties of such systems. These averaged properties are usually calculated under the assumption that the systems will pass through "all possible states", given the energy or the temperature in question (we do not even try to formulate this rigorously). Though this assumption has not yet been justified in general, the problem has led to the development of a theory of the averaged properties of solutions of dynamical systems, now known as ergodic theory.

In passing we observe that the partially positive answer to the problem of the stability of the solar system, as it was given by the K.A.M. (Kolmogorov, Arnold, Moser) theory, shows that the ergodicity postulate is to a certain extent wrong.

Returning to the volume under discussion, I mention the following aspects. Though the authors avoid references to examples outside the *mathematical* theory, the mathematical examples play a prominent role, which I found very attractive. In the introduction there is a carefully chosen set of examples, ranging from simple ones like linear maps and gradient flows to hyperbolic automorphisms and symbolic dynamical systems, which is used to illustrate basic notions like stability, various asymptotic topological invariants, and notions from ergodic theory.

Also, in the more advanced part 4 on hyperbolic dynamics there is a beautiful collection of examples, some, like the ones on rank-one symmetric spaces, recently found. These examples, apart from their intrinsic interest, will probably help the student who wants to learn the subject from this book.

In those parts where the authors have been less active as researchers there are some inaccuracies. The ones which I found were in the notes on Chapter 7, Section 2 (extensions of the Kupka-Smale theorem), where results on the generic occurrence of homoclinic points in symplectic and volume-preserving maps are incorrectly summarized, and in the proof of Proposition 7.3.3 (structural stability of the saddle node), where the continuity of the conjugating homeomorphism, as a function of the parameter, is missed—actually this is a subtle point which has been overlooked also by other authors.

This remark should not be interpreted in the sense that I consider the work not carefully written: in a volume of about 800 pages it is hard, if not impossible, not to miss a few points, and the points mentioned are rather the specialty of the reviewer than of the authors of this volume.

The treatment of hyperbolic systems, including their ergodic properties, mainly in the fourth part, with preliminaries in parts 1 and 2 and with an extension in the supplement, is in my opinion really excellent. It is the most accessible treatment of this theory, which combines the analytic-geometric approach, initiated by Smale and others, with the analytic-probabilistic approach, initiated by Kolmogorov, Sinai and others. The authors show the wisdom of not trying to be complete or extremely compact. The theory is still treated in such a depth that, for example, the deep results of Margulis on closed geodesics on manifolds with negative sectional curvature could be covered.

Finally, I would like to mention the high quality of the illustrations. My favorites are the homoclinic web on page 276 and the Smale attractor on page 533.

Floris Takens
University of Groningen