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Methods of noncommutative analysis, by V. Nazaikinskii, V. Shatalov, and B. Sternin, de Gruyter Studies in Math., vol. 22, de Gruyter, 1996, x+373 pp., DM 198, ISBN 3-11-014732-0

An important part of operator theory is the study of functions of operators. In the case of a single linear operator A, there is the Dunford operator calculus,

$$f(A) = \frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1}\, dz, \tag{1}$$

defined when f is holomorphic on a region in $\mathbb{C}$ containing the spectrum of A, with boundary γ. If A is self-adjoint, there is the functional calculus arising from the spectral decomposition $dE(\lambda)$. For a Borel function f defined on $\mathbb{R}$,

$$f(A) = \int f(\lambda)\, dE(\lambda). \tag{2}$$

Also, for self-adjoint A, one has (for well-behaved f)

$$f(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) e^{itA}\, dt, \tag{3}$$

by the Fourier inversion formula plus the spectral theorem, where

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int f(\lambda) e^{-i\lambda t}\, d\lambda.$$

If one has an n-tuple of noncommuting operators $A = (A_1, \dots, A_n)$, there are a number of different ways to define functions of these operators. One method, explored in this book, is a "Feynman-ordered" functional calculus. The general setup is the following. Let $\mathcal{A}$ be a topological algebra of operators, $\mathcal{F}$ a topological algebra of functions of one variable. An element $A_j \in \mathcal{A}$ is called an $\mathcal{F}$-generator if $f \mapsto f(A_j)$ defines a continuous homomorphism $\mu_{A_j} : \mathcal{F} \to \mathcal{A}$. If $A_1, \dots, A_n$ are $\mathcal{F}$-generators, one has an n-linear map

$$\mu_A : \mathcal{F} \times \cdots \times \mathcal{F} \to \mathcal{A}, \qquad \mu_A(f_1, \dots, f_n) = f_n(A_n) \cdots f_1(A_1). \tag{4}$$

This extends to the completed projective tensor product

$$\mathcal{F}_n = \mathcal{F} \hat{\otimes} \cdots \hat{\otimes} \mathcal{F} \text{ (n factors)}, \tag{5}$$

under appropriate topological hypotheses, producing

$$f({}^1A, \dots, {}^nA_n) = \mu_A(f),\ f \in \mathcal{F}_n. \tag{6}$$

One calls $({}^1A_1, \dots, {}^nA_n)$ a Feynman tuple. Here the overset indices indicate the order in which the operators are applied. For example, if A and B are $\mathcal{F}$-generators and $f_j \in \mathcal{F}$, $g(x, y) = f_1(x) f_2(y)$, then

$$g({}^1A, {}^2B) = f_2(B) f_1(A), \qquad g({}^2A, {}^1B) = f_1(A) f_2(B). \tag{7}$$

Thus, if $e^x \in \mathcal{F}$,

$$e^{({}^1A + {}^2B)} = e^B e^A, \qquad e^{({}^2A + {}^1B)} = e^A e^B. \tag{8}$$

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By comparison with (2) and (3) we have, for self-adjoint n-tuples and appropriate functions f,

$$f({}^1A_1, \dots, {}^nA_n) = \int f(\lambda_1, \dots, \lambda_n)\, dE_n(\lambda_n) \dots dE_1(\lambda_1), \tag{9}$$

where dE_j is the spectral measure of A_j. Also,

$$f({}^1A_1, \dots, {}^nA_n) = (2\pi)^{-n/2} \int \hat{f}(t_1, \dots, t_n) e^{it_nA_n} \cdots e^{it_1A_1}\, dt_1 \cdots dt_n. \tag{10}$$

An ordered operator calculus of this nature was discussed by V. Maslov in [Mas] and applied to problems in asymptotic analysis. There has been further work on such operator calculus by Maslov and a number of colleagues. The book currently under review has the goal to present an accessible introduction to the subject.

To give some of the flavor of the use of the ordered operator calculus presented in this book, we consider the action of derivations on $f(A)$. Assume $A \in \mathcal{A}$ is an $\mathcal{F}$-generator, and assume there is a continuous map

$$\frac{\delta}{\delta x} : \mathcal{F} \to \mathcal{F}_2, \qquad \frac{\delta f}{\delta x}(x, y) = \frac{f(x) - f(y)}{x - y}, \tag{11}$$

this function given by $f'(x)$ on the diagonal. To begin, we look at the commutator $[B, f(A)]$, given $B \in \mathcal{A}$. Then

$$\begin{aligned} [B, f(A)] &= {}^2B(f({}^1A)) - f({}^3A)) = {}^2B({}^1A - {}^3A) \frac{\delta f}{\delta x}({}^1A, {}^3A) \\ &= {}^2B({}^1A - {}^3A) \frac{\delta f}{\delta x}({}^0A, {}^4A) = {}^2[B, A] \frac{\delta f}{\delta x}({}^0A, {}^4A) \\ &= {}^2[B, A] \frac{\delta f}{\delta x}({}^1A, {}^3A). \end{aligned} \tag{12}$$

If one passes to the left-regular representation of $\mathcal{A}$ on itself, then any derivation D on $\mathcal{A}$ is represented by a commutator on $\mathrm{End}(\mathcal{A})$, so one obtains

$$Df(A) = {}^2[DA] \frac{\delta f}{\delta x}({}^1A, {}^3A). \tag{13}$$

Now, if we expand our considerations to have $\mathcal{A}$ consist of operator-valued functions of t and consider the derivation $d/dt : \mathcal{A} \to \mathcal{A}$, then we obtain the formula

$$\frac{d}{dt} f(A(t)) = {}^2A'(t) \frac{\delta f}{\delta x}({}^1A(t), {}^3A(t)). \tag{14}$$

For example, if $e^x \in \mathcal{F}$,

$$\frac{d}{dt} e^{A(t)} = {}^2A'(t) \frac{e^{{}^3A(t)} - e^{{}^1A(t)}}{{}^3A(t) - {}^1A(t)}. \tag{15}$$

We can compare (15) with a common formula, derived as follows. The left side of (15) is $M(1, t)$, where $M(s, t) = (d/dt)e^{sA(t)}$. Note that

$$\frac{\partial M}{\partial s} = A(t)M(s, t) + A'(t)e^{sA(t)}, \ M(0, t) = 0. \tag{16}$$

We can solve (16) by Duhamel's principle, obtaining

$$M(s, t) = \int_0^s e^{(s-\sigma)A(t)} A'(t) e^{\sigma A(t)}\, d\sigma. \tag{17}$$

Using the identity

$$e^{\sigma A}A'e^{-\sigma A} = e^{\sigma \operatorname{ad} A}A', \tag{18}$$

we get

$$\frac{d}{dt}e^{A(t)} = \varphi(\operatorname{ad} A(t))(A'(t)) \cdot e^{A(t)},\ \varphi(z) = \frac{e^z - 1}{z}. \tag{19}$$

In the current book the formula (15) is applied to derive an elegant form of the Campbell-Hausdorff formula, for

$$e^C = e^B e^A, \tag{20}$$

representing C in terms of A and B, and commutators. The method is to derive a formula for $D'(t)$ when $D(t)$ satisfies $e^{D(t)} = e^{tB}e^A$. Making use of (15), the authors obtain the equation

$$D'(t) = \psi(\operatorname{ad} D(t))(B),\ \psi(z) = \frac{z}{e^z - 1}, \tag{21}$$

a result that also follows directly from (19) (this sort of derivation of the Campbell-Hausdorff formula from (19) is given in [HS]). Noting that $e^{\operatorname{ad} D(t)} = e^{t \operatorname{ad} B}e^{\operatorname{ad} A}$, one obtains

$$D'(t) = \Phi(e^{t \operatorname{ad} B}e^{\operatorname{ad} A})(B),\ \Phi(\zeta) = \frac{\log \zeta}{\zeta - 1}, \tag{22}$$

and hence, if $\| \operatorname{ad} A\|$ and $\| \operatorname{ad} B\|$ satisfy convenient bounds,

$$C = A + \int_0^1 \Phi(e^{t \operatorname{ad} B}e^{\operatorname{ad} A})B\, dt. \tag{23}$$

This version of the Campbell-Hausdorff formula is very nice. It ought to be given in standard books on the theory of Lie groups, but [HS] is the only such reference I know that has it.

An important example of ordered functional calculus arises when

$$A = (X_1, \ldots, X_n, D_1, \ldots, D_n), \qquad X_j f(x) = x_j f(x), \qquad D_j f(x) = -i\frac{\partial f}{\partial x_j},$$

for a function f on $\mathbb{R}^n$. Then, a calculation using (10) yields

$$a(\overset{2}{X}, \overset{1}{D})f(x) = (2\pi)^{-n/2} \int a(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{24}$$

a common representation for pseudodifferential operators. A variant is

$$b(\overset{3}{X}, \overset{1}{X}, \overset{2}{D})f(x) = (2\pi)^{-n} \iint b(x,y,\xi)f(y)e^{i(x-y)\cdot\xi}\, dy\, d\xi. \tag{25}$$

In these formulas, following a standard convention, we allow overset indices to coincide for commuting operators, such as X_j, X_k. Analysis of pseudodifferential operators played a major role in [Mas] and also in the current book.

It is important to understand compositions of pseudodifferential operators. More generally, one can try to represent products of functions of a Feynman tuple $A = (\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ as another operator in that form, i.e., given $f, g \in \mathcal{F}_n$, to find $h \in \mathcal{F}_n$ such that

$$f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)g(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) = h(\overset{1}{A}_1, \ldots, \overset{n}{A}_n). \tag{26}$$

The book relates this problem to the construction of a left-ordered representation of A, defined to be a Feynman tuple $L = ({}^1L_1, \dots, {}^nL_n)$ of operators $L_j : \mathcal{F}_n \to \mathcal{F}_n$, $1 \le j \le n$, satisfying the following two conditions. First, for any $f \in \mathcal{F}_n$, $1 \le j \le n$,

$$ {}^{n+1}A_j f({}^1A_1, \dots, {}^nA_n) = (L_j f)({}^1A_n, \dots, {}^nA_n). \tag{27} $$

Second, for $f \in \mathcal{F}_n$,

$$ f = f(y_1, \dots, y_j) \Rightarrow L_j f = y_j f. \tag{28} $$

In such a case, (26) holds, with

$$ h = f * g = f(L)g. \tag{29} $$

With this product, $(\mathcal{F}_n, *)$ is an algebra, which might not be associative. The condition for associativity is that $(f * g)(L) = f(L)g(L)$.

For the pseudodifferential operator calculus (23) one has a left-ordered representation:

$$ A_1 = -i\frac{\partial}{\partial x},\ A_2 = x; \qquad L_1 = \xi - i\frac{\partial}{\partial x},\ L_2 = x. \tag{30} $$

The condition for associativity is satisfied in this case.

There are related problems, such as expressing functions of creation and annihilation operators in Wick normal form and studying functions of an n-tuple of operators spanning a Lie algebra, especially in the nilpotent case, considered in the book. There are also brief discussions of other algebras, such as those satisfying perturbed Heisenberg relations and Fadeev-Zamolodchikov algebras. In the latter case, there is a discussion of relations between the associativity criterion for $(\mathcal{F}_n, *)$ (when $\mathcal{F}$ is the algebra of polynomials and the product $*$ is determined by a left-ordered representation), the Poincáre-Birkhoff-Witt property, and the Yang-Baxter equation.

The third chapter of this four-chapter book is devoted to constructing approximate solutions to partial differential equations. The development of such applications was the main point of the work in [Mas]. The current book devotes less space to these applications, consistent with its introductory nature, but the problems considered provide a good illustration of the relevant techniques. Examples include difference-scheme approximations, hyperbolic equations with coefficients growing at infinity, and subelliptic operators with double characteristics. We will describe in a little more detail another example considered there.

This is the problem of the asymptotic behavior of solutions to

$$ \begin{gathered} -\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^2 u}{\partial x^2} - \lambda^2 b(x)^2 u = \lambda\delta(x)H(t)r(t)e^{-i\lambda q(t)}, \\ u(t,x) = 0, \quad \text{for } t < 0. \end{gathered} \tag{31} $$

Here, $H(t) = 0$ for $t < 0, 1$ for $t > 0$, and $r(t)$ and $q(t)$ are given smooth real-valued functions. This equation arises as a model for propagation of electromagnetic waves in a plasma. One wants to produce an approximation whose error is small as $\lambda \to \infty$ when measured in a norm incorporating a large number of derivatives. This is an interesting problem, exhibiting several subtle phenomena. Here we sketch an alternative approach to the analysis, emphasing geometrical phenomena. We will make use of some concepts of microlocal analysis which we cannot take the space here to introduce but which are explained in [H] and in [GU] and [MU].

Let us consider the partial inverse Fourier transform with respect to λ:

$$w(t,x,y) = \frac{1}{\sqrt{2\pi}} \int u(t,x,\lambda) e^{i\lambda y}\, d\lambda. \tag{32}$$

Then w solves

$$\begin{gathered} -\frac{\partial^2 w}{\partial t^2} + c^2 \frac{\partial^2 w}{\partial x^2} + b(x)^2 \frac{\partial^2 w}{\partial y^2} = iH(t) r(t) \delta(x) \delta'(y - q(t)), \\ w(t,x,y) = 0, \quad \text{for } t < 0. \end{gathered} \tag{33}$$

Note that w has compact support in (x,y)-space, for t in any bounded interval $[0,T]$. Hence, approximation of w modulo a smooth error is effective in approximating

$$u(t,x,\lambda) = \frac{1}{\sqrt{2\pi}} \int w(t,x,y) e^{-i\lambda y}\, dy, \tag{34}$$

in the sense described above.

Note that the right side $\varphi(t,x,y)$ of (33) is a distribution supported on the curve $y = q(t)$, $x = 0$, $t \geq 0$. The following feature plays a major role in the nature of the solution. The wave front set of φ consists both of all nonzero cotangent vectors in $T^*(\mathbb{R}^3)$ lying over $(0,0,0)$ and all nonzero vectors conormal to the curve described above. More precisely, φ is a distribution associated to the union of these two transversally intersecting Lagrangians, in the sense of [MU]. The wave front set of the solution to (33) will lie in the union of the null bicharacteristics of the linear hyperbolic operator

$$P = -\partial_t^2 + c^2 \partial_x^2 + b(x)^2 \partial_y^2, \tag{35}$$

going forward in time from points in $WF(\varphi)$. In particular, there will be the forward light cone, swept out by rays passing over the origin. This gives rise to the "transient" part of the solution. Whether there are null bicharacteristics passing over a point on the curve $y = q(t)$, $x = 0$, $t > 0$ depends on the sign of $q'(t) - b(0)$. Singularities propagate off such a point only if $q'(t) - b(t) \geq 0$. If $q'(t) > b(0)$ for all t, then $w|_{\{t>0\}}$ is a distribution associated to a union of Lagrangians, which intersect at most pairwise and transversally. Its symbol can be computed by the methods of [MU] and [GU].

Figure 1 shows the singular support of $w(t) \in \mathcal{E}'(\mathbb{R}^2)$, for some fixed $t > 0$. In the case depicted there, we take $b(x) = b$ to be constant and $q'(t)$ to be a constant $> b$. The singularity of $w(t)$ is $\frac{1}{2}$-unit stronger on the line segments in Figure 1 than it is on the ellipse (away from the points of intersection). If $b(x)$ and $q'(t)$ vary and $q'(t) > b(0)$ for all t, one would get a curvy variant of this picture, for small $t > 0$, generally with subsequent formation of caustics. If $q'(t) - b(0)$ were to change sign, other phenomena would arise.

In keeping with the goal to present a self-contained introduction to the subject, the authors present some relevant functional-analytic background, mostly collected in the final chapter. They do a fairly good job, though we mention one point where the foundational material might have been more completely clarified by slightly extending the framework for functional calculus given by (4), (5), and (6).

Motivation to do this can be seen by studying the formula (12). Note that the third quantity in (12) has the form

$$g({}^1A, {}^2B, {}^3A), \qquad g(x,y,z) = y(x-z)\frac{\delta f}{\delta x}(x,z). \tag{36}$$

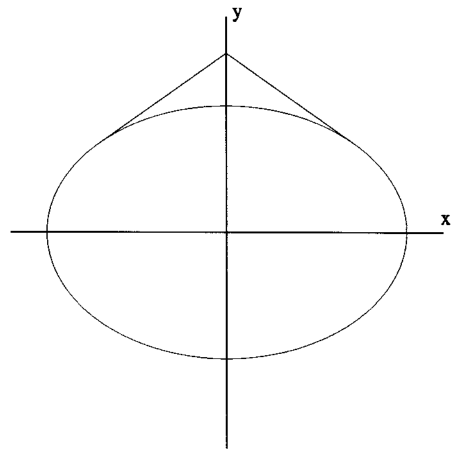

FIGURE 1

Now, the functional calculus was said to be defined when g belongs to $\mathcal{F}_n$ (here, $n = 3$) and the arguments are each $\mathcal{F}$-generators. However, while A was assumed to be an $\mathcal{F}$-generator, B was not. One could add the hypothesis that B is also an $\mathcal{F}$-generator, but that would be inconvenient in view of applications to (13) and (14), and in fact it is not necessary. Note that in (36) $g(x, y, z)$ has the special property of being linear in y. A natural framework for the functional calculus that is literally applicable to (12) is the following. Instead of a single topological algebra $\mathcal{F}$ of functions of one variable, one might have a sequence of such algebras $\mathcal{F}^\nu$ and continuous homomorphisms $\mathcal{F}^\nu \to \mathcal{A}, f \mapsto f(A)$; some of the algebras $\mathcal{F}^\nu$ could coincide. Let A_ν be $\mathcal{F}^\nu$-generators. Then we have a multilinear map

$$\mu_A : \mathcal{F}^1 \times \cdots \times \mathcal{F}^n \to \mathcal{A}, \qquad \mu_A(f_1, \ldots, f_n) = f_n(A_n) \cdots f_1(A_1), \tag{37}$$

which extends to

$$\mu_A : \mathcal{F}^1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}^n \to \mathcal{A}, \tag{38}$$

under such topological hypotheses as used by the authors, yielding

$$f({}^1A_1, \ldots, {}^nA_n) = \mu_A(f), \;\; f \in \mathcal{F}^1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}^n. \tag{39}$$

Thus in (36) we can take $\mathcal{F}^1 = \mathcal{F}^3 = \mathcal{F}$ and $\mathcal{F}^2 = \mathcal{P}$, the algebra of polynomials in one variable, with its natural inductive limit topology.

Another extension of the ordered operator calculus, mentioned but not systematically developed in the book, involves continuous-order parameters. For example, the solution to the system

$$\frac{du}{dt} = A(t)u, \;\; u(0) = y, \tag{40}$$

which is

$$u(t) = \lim_{n\to\infty} e^{A(t)/n} e^{A(t(n-1)/n)/n} \cdots e^{A(t/n)/n} y, \tag{41}$$

can also be written, with obvious notation, as

$$u(t) = e^{\int_0^t {}^{\tau}A(\tau)\, d\tau} y. \tag{42}$$

To take a variant,

$$\frac{du}{dt} = [A(t) + B(t)]u, \ u(0) = y, \tag{43}$$

appealing to a generalized Trotter product formula when appropriate, we can write the solution as

$$u(t) = \lim_{n\to\infty} e^{A(t)/n} e^{B(t)/n} \cdots e^{A(t/n)/n} e^{B(t/n)/n} y, \tag{44}$$

and hence write

$$u(t) = e^{\int_0^t [{}^{\tau}A(\tau) + {}^{\tau-0}B(\tau)]\, d\tau} y. \tag{45}$$

Indeed, the work of Feynman on ordered operator calculus involved such "time-ordered" operators. If in (43) we have $A(t) = \Delta$, the Laplace operator, and $B(t) = -V(t, x)$, multiplication by a t-dependent potential, operating on functions $u(t, x)$, then an evaluation of (44) leads to Kac's formula for the solution to (43) in terms of an integral over path-space with respect to Wiener measure. This work followed Feynman's analysis of the analogous Schrödinger equation, where $A(t) = i\Delta$, $B(t) = -iV(t, x)$, which led to the somewhat more recondite Feynman path integral.

So, while there is much that remains to be done on ordered operator calculus, there is much that has been done. The book by Nazaikinskii, Shatalov, and Sternin can be recommended as a good place to start the study of this subject.

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