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Random series and stochastic integrals: Single and multiple, by Stanislaw Kwapien and Wojbor A. Woyczynski, Probability and Its Applications, Birkhäuser, Basel and Boston, MA, 1992, xvi+360 pp., ISBN 0-8176-3572-6

This book is more a research monograph than a graduate text and provides a mine of information that is not to be found in the usual books on probability theory or stochastic processes. Most of the material has appeared in the recent papers of the authors themselves and of other workers in the field.

As the title suggests, there is an attempt to treat a part of the theory of stochastic processes (stochastic integrals) by basing it on what Paul Lévy called "probabilités dénombrables". The approach itself is by no means new, since, in fact, most modern treatments of Brownian Motion introduce it as the limit of a uniformly convergent (almost sure) random series. This might be the place to point out that the proof of the above fact given in Chapter 2 rests on a "functional-analytic" argument and not on the easiest and most natural method (due, I believe, to Ciesielski) which uses the Haar basis. The latter method has the further advantage that it extends immediately to the definition of the Brownian sheet and indeed of the Wiener process in several parameters. On the other hand, as shown in a remark in the book, the fractional differentiation operator technique can be extended to yield the γ-Hölder continuity ($\gamma < \frac{1}{2}$) of the Brownian motion sample paths.

For the convenience of the nonspecialist mathematician, it might be desirable to sketch here a very brief background of probability theory and stochastic processes, relevant to the subject matter of the book. When the calculus of random variables began to be developed in the '30s and '40s (a random variable (r.v.) is, for now, a real-valued measurable function on a measure space endowed with a probability measure), the two most important limit theorems studied were the following:

1. Convergence in distribution (or in law). Consider the sequence of normed sums of independent, identically distributed (i.i.d.) r.v.'s $X_1, X_2, \dots, X_n$, $(X_1 + \cdots + X_n - b_n/a_n)(a_n > 0$ and b_n real constants). What are the limit laws or distribution functions F such that

$$\text{Prob}\left\{\frac{X_1 + \cdots + X_n - b_n}{a_n} \leq x\right\} \to F(x) \tag{1}$$

for every continuity point x of F, as $n \to \infty$?

The problem has not been stated in the utmost generality but is general enough for present purposes. If the common variance is finite, then a result which practically goes back to Laplace and in special cases to de Moivre (a contemporary of Newton) says that (1) holds with $b_n = nE(X_1)$, $a_n = \sqrt{n \text{ variance}(X_1)}$ and

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du.$$

The complete answer to problem (1), given in the '20s and '30s of this century, says that the only possible limit laws F are stable laws. If Y is a symmetric random variable and has a stable distribution, then its characteristic function (Fourier

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transform) is given by $Ee^{ivY} = e^{-c|v|^\alpha}$ where $0 < \alpha \leq 2$ and c is a positive constant. The normal or Gaussian distribution corresponds to $\alpha = 2$.

2. The second problem is concerned with the convergence of infinite series of independent random variables—convergence in probability, almost sure (i.e., a.e. with respect to the probability measure P), or L^p-convergence. The early work of Kolmogorov and Khinchine was extended much later to series of Banach-space valued r.v.'s by Itô and Nisio.

The theory of stochastic integrals is a relatively new part of the theory of stochastic processes, the study of families of random variables indexed by a continuous parameter or, one might say, of random variables taking values in infinite-dimensional spaces. One such random variable which made its entry rather early in the history of the subject is the Brownian motion (also called the Wiener process after Wiener's work). After several of its properties had been heuristically but brilliantly derived and applied to financial problems by Bachelier (1900) and to statistical physics by Einstein (1905), it was Wiener in 1923 who defined the probability measure (denoted here by μ_w) on the space C of continuous functions on R_+ which determines the distribution of the Wiener process given on the probability space $(C, \mathcal{B}, \mu_w)$, ($\mathcal{B}(C)$ being the Borel sets in C).

Stochastic integrals of different kinds have been defined (and not only with respect to the Wiener process) to deal with practical problems, notably continuous-time linear prediction and filtering. These integrals are the continuous analogs of linear functions or forms of random variables. In order to solve more general nonlinear estimation and prediction problems as well as being motivated by the desire to construct a satisfactory mathematical theory of functionals of Brownian motion, Wiener introduced the multiple stochastic integral in his famous 1938 paper on homogeneous chaos. His work was modified and completed by Itô, who supplied the essential step in the development of an orthogonal chaos expansion for every square integrable "Wiener" functional in terms of multiple Wiener-Itô integrals. Itô also showed the equivalence of his "stochastic" approach with the purely analytical expansion in terms of Fourier-Hermite functionals obtained independently by Cameron and Martin. (As far as I understand it, the word "chaos" was used by Wiener to indicate the chaotic influence of Brownian motion on physical phenomena.) Hence this entire corpus of work may justly be regarded as the foundation of nonlinear stochastic analysis.

The study of polynomial or multilinear forms of random variables undertaken in this book is a natural and logical precursor to the attempt in the second half of this book to generalize the Wiener-Itô-Cameron-Martin theory to processes other than the Brownian motion. The topics in the book are organized as follows.

The first part, on random series, consists of the following chapters:

1. Basic inequalities for random linear forms in independent random variables
2. Convergence of series of independent random variables
3. Domination principles and comparison of sums of independent random variables
4. Martingales
5. Domination principles for martingales
6. Random multilinear forms in independent random variables and polynomial chaos

Stochastic integrals are treated in the following chapters of the second part:

7. Integration with respect to general stochastic measures
8. Deterministic integrands
9. Predictable integrands
10. Multiple stochastic integrals

A preliminary chapter introduces, besides standard topological and measure-theoretic notions, an extension of Orlicz spaces called Musielak-Orlicz spaces (which play a key role in characterizing a.s. convergence of sequences of independent random variables) and also the specially important random variables—Bernoulli, Gaussian and α-stable sequences. There are two appendices—one on vector measures—which should be particularly convenient and useful to graduate students. The other appendix has the title "Unconditional and bounded multiplier convergence of random series".

The most valuable portions of the book, in my view, are (i) the inequalities which form a prominent part of the first six chapters, (ii) the careful and detailed discussion of stochastic measures leading to stochastic integrals with respect to semimartingales, and (iii) the results on stable processes. I believe that most if not all the material in these three topics appears for the first time in a book, so that these topics are no longer in the frontiers of research but are brought into the mainstream of probability theory and stochastic processes. In achieving this the authors have made an important contribution to the dissemination of some of the modern developments of the subject.

In (i) one should especially mention the tail inequalities, hypercontractive and decoupling inequalities. These are proved not only for linear forms of random variables but also for multilinear forms. The idea of decoupling is particularly useful in simplifying proofs. It is simply explained in the book for quadratic forms as follows: Let

$$Q(X_1,\dots,X_n) = \sum_{1\le i\le j\le n} a_{ij}X_iX_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{j-1} a_{ij}X_i\right)X_j$$

be a homogeneous random quadratic form with a_{ij} constants belonging to a Hilbert space. If $(X_1',\dots,X_n')$ is an independent copy of the symmetric random vector $(X_1,\dots,X_n)$, then

$$Q'(X_1,\dots,X_n,X_1',\dots,X_n') = \sum_{j=1}^{n}\left(\sum_{i=1}^{j-1} a_{ij}X_i\right)X_j'$$

is the sum of a "decoupled" sequence $(\sum_{i=1}^{j-1} a_{ij}X_i)X_j'$. $j = 1,\dots,n$ which is "tangent" to the sequence $(\sum_{i=1}^{j-1} a_{ij}X_i)X_j$ relative to the sigma fields $\mathcal{F}_j = \sigma(X_1,\dots,X_j,X_1',\dots,X_j')$. The simplification consists in reducing the proof of the desired inequality for a given multilinear form to a corresponding problem for a lower-degree polynomial.

Chapter 6 also contains the inequalities for random multilinear forms (or polynomial chaos) where most of the above techniques are brought into play.

Exponential moment bounds for Banach space-valued Gaussian random variables are obtained in several different ways in Chapters 2 and 3. Fernique's bound is first given along with his proof. Using the hypercontraction principle, a strong exponential moment bound is given for a Banach space-valued Gaussian series $S = \sum_1^\infty a_iX_i$, a_i's chosen from a Banach space, and X_i canonical Gaussian variables. A more precise estimate for $E \exp \alpha \|S\|^2$ is given later in the chapter with an improved range for α.

The almost-sure convergence of the series $S = \sum_{i=1}^{\infty} a_i X_i$ (defined as above) is a necessary result before one can discuss the bounds for its moments. This convergence result proved in Chapter 2 is referred to as a "Karhunen-Loève" representation of a Gaussian measure. The same label is attached to a similar almost-sure result for the Wiener chaos in Chapter 6. The names should mystify the modern graduate student and indeed any research worker who has not had the standard graduate course on advanced probability that the authors (and the reviewer) had in their student days. The reference is to the series representation (perhaps the first of its kind) of a continuous Gaussian process in terms of the spectral expansion of its covariance function. Incidentally, as long as we are mentioning authors, the original K-L theorem should be properly called the Karhunen-Kosambi-Loève theorem. The result was independently discovered during World War II by the Indian mathematician and Marxist historian D. D. Kosambi.

The part devoted to stochastic integrals gives a detailed treatment of stochastic measures (supplemented by a helpful appendix on vector measures) and stochastic integrals with respect to semimartingales. Decoupling inequalities developed earlier are used to give a simpler construction of stochastic integrals for the special case when the integrator is a process of independent increments.

Although the main thrust of the second part is the definition and properties of multiple stochastic integrals with respect to processes of independent increments (or independently scattered measures), it has to be recognized, as the authors do, that no workable definition of multiple stochastic integrals of order $d (d > 2)$ has so far been possible (with the partial exception of α-stable processes) at this level of generality.

The Wiener chaos decomposition of the L^2-space of Wiener functionals is introduced in Chapter 6 in a "denumerable" fashion in keeping with the general philosophy of the book. The aim is to obtain the closure in L^2 of $\mathcal{K}_d^0$, the family of all (real) polynomial chaoses of order d based on a given sequence $\gamma_1, \gamma_2, \ldots$ of i.i.d. standard Gaussian random variables. By introducing the problem in this way, Wiener's nonlinear analysis of Brownian functionals is made a part of the general study of polynomial chaoses undertaken in the book. The main result, which uses Hermite polynomials, is due to Cameron and Martin; there is no mention of this either in the text or in the notes at the end of the chapter, although their basic paper is cited in the references. (It should perhaps be pointed out that in deriving the properties of Hermite polynomials, on pages 176-177, completeness is left out.) Itô's equivalent decomposition in terms of multiple Wiener-Itô integrals is given later in Chapter 10, where multiple stochastic integrals are introduced.

The difficulties inherent in the attempt to extend the Wiener-Itô theory of multiple integrals to integrators X which are arbitrary processes of independent increments can be visualized in the work of Chapter 10. A characterization of a symmetric function $f : T^2 \to R$ in order that $\int f \, dX \, dX$ be defined is given in terms of a function Ψ related to the modular Φ and the control function ν of X. The most interesting consequence is the application to the case when X is a symmetric, stationary α-stable process, given merely as an example. For higher-order $(d > 2)$ multiple integrals, a sufficient condition for $f : T^d \to R$ to be integrable is given without proof at the end of the chapter.

On the other hand, for the Brownian integrator the Wiener-Itô theory has been enriched by the further development of stochastic integration theory. Only a reluctance to accord Brownian motion the importance it deserves can possibly explain the omission of some of these recent advances. One example of this omission is the Stratonovich integral, single and multiple. In view of the strong emphasis in the book on the role of multilinear forms in nonlinear random theory, it may be stated here that multilinear forms $\sum_{1\leq i_1\leq\cdots\leq i_d\leq n} a_{i_1\cdots i_d}X_{i_1}\cdots X_{i_d}$ where the coefficients $a_{i_1\cdots i_d}$ are *not* put equal to zero if two or more indices are equal are the "building blocks" of multiple Stratonovich integrals. A brief reference to one of the anticipative stochastic integrals—for instance, the Skorokhod integral whose definition can be based on the multiple Itô integral—would have brought this part of the book more in line with modern developments.

One wishes that greater prominence had been given to the generalizations of some of the results on stable processes, perhaps by collecting them in a separate chapter (although I recognize the organizational difficulty in doing so). In the book they appear as special examples. For instance, necessary and sufficient conditions for the almost-sure convergence of the series $\sum_{i=1}^{\infty} a_i X_i$, where $a_i \in L^p(T, \mathcal{A}, \mu)$ and X_i are i.i.d. α-stable r.v.'s, $0 < \alpha < 2$, are given as a corollary of a more general "three series" theorem in Chapter 2. Quadratic chaos for the stable case is similarly relegated to a corollary in Chapter 6. Since non-Gaussian stable processes form, next to Brownian motion, a distinguished family of stochastic processes, these results together with their discrete versions appearing earlier in the book considerably enhance the value of the book.

In a book in which the authors are at pains to work with Banach space-valued random variables, it is curious that there is no mention of abstract Wiener spaces and theorems relating to them. Another omission—more understandable because the subject has grown so vast that no present-day book on stochastic processes can afford to be self-contained—is the absence of reference to probability measures on infinite-dimensional spaces, e.g., Banach and Hilbert spaces. Graduate students in "probability theory, stochastic processes and theoretical statistics" (quoted from the introduction) for whom this book is also intended could find the lack of this basic information a handicap in mastering the material. How would they know the distinction between, say, an n-dimensional Gaussian random variable and a Gaussian variable taking values in an infinite-dimensional Hilbert space? The book could have easily supplied the necessary details by judiciously amplifying the chapter on preliminaries (Chapter 0). As it is, the bare definition of a Gaussian probability measure on a separable Banach space F given in subsection 0.6 does not tell the whole story. The reader might be left with the impression that any bilinear form $V(x', y')$ can serve as the covariance of a Gaussian measure. That such is not the case can be seen by taking F to be an infinite-dimensional Hilbert space and $V(x', y')$ to be the inner product of x' and y'.

The authors' claim in the introduction that the book can serve as "a foundation for the stochastic Itô calculus and the theory of stochastic differential equations" is hard to justify in a book where there is no mention, even in a disguised form, of the Itô formula for semimartingales—a fundamental result without which one cannot take the first step in stochastic calculus. Indeed, there is no need for the authors to make such a claim, for even without it the book is of considerable value to the aspiring research student, as I have pointed out earlier in this review.

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