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Diffusions, Markov processes, and martingales, Volume One: Foundations, Second Edition, by L. C. G. Rogers and D. Williams, John Wiley & Sons, Chichester, West Sussex, England, 1994, xx + 386 pp., \$79.95, ISBN 0-471-95061-0

Diffusions, martingales, and Markov processes are each particular types of stochastic processes. A stochastic process, in a state space E, with parameter set $\mathbf{T}$, is a family $(X_t)_{t\in\mathbf{T}}$ of E-valued random variables, or equivalently, a random variable X that takes its values in a space of functions from $\mathbf{T}$ to E. Usually, the parameter set $\mathbf{T}$ is a subset of $\mathbf{R}$, often $[0,\infty)$ or $\{0,1,2,3,\dots\}$, the parameter is thought of as time, and the functions from $\mathbf{T}$ to E are thought of as paths in E. A stochastic process thus describes the evolution in time of a system for which we do not know the path in state space that the system will follow, but only the probability (usually "infinitesimal") of each possible path that the system might follow. A Markov process is a stochastic process whose future evolution at any given time t depends only on the state of the system at the present time t and not on the states of the system at past times $s < t$. A diffusion is a Markov process whose paths are continuous functions of time. Brownian motion is the quintessential example of a diffusion, and the Poisson process is the quintessential example of a Markov process that is not a diffusion. A martingale is a stochastic process that models the fortune of a gambler as a function of time if the gambler is playing a fair game.[1] Martingale theory turns out to be a powerful tool for the study of Markov processes, because a Markov process has many martingales that are naturally associated to it.

The evolution in time of a Markov process in E with time set $[0,\infty)$ may be described by a *transition function* $P_{s,t}(x,dy)$, $0 \le s < t < \infty$, $x, y \in E$, which gives the conditional probability that $X_t \in dy$ given that $X_s = x$. By considering the *space-time* process $t \mapsto (X_t, t)$, one may reduce to the case where $P_{s,s+t}$ does not depend on s. Hence it is usual to assume that one is in this *time-homogeneous* case and to write P_t for $P_{s,s+t}$. When $E = \mathbf{R}^d$, the special case where P_t is also *spatially homogeneous* is fundamental. In this case, $P_t(x,dy) = P_t(z+x, z+dy)$, and the family $\mu_t(dx) = P_t(0,dx)$, $0 < t < \infty$, is a convolution semigroup of probability measures on $\mathbf{R}^d$. To avoid uninteresting pathology, it is customary to assume that μ_t converges weakly to the unit point mass at the origin as t tends to $0+$. The process X is then called a Lévy process. In the theory of stochastic differential equations, the Lévy processes play a role analogous to the role played by straight lines in ordinary calculus.[2] In the case of Brownian motion in $\mathbf{R}^d$, $\mu_t(dx)$ has a normal density $(2\pi t)^{-d/2}\exp\{-|x|^2/(2t)\}$ with respect to Lebesgue measure and X can be taken to have continuous paths. Conversely, if X is a Lévy process in $\mathbf{R}^d$ with continuous paths, then for each t, since the random variable $X_t - X_0$ is the sum of the independent increments $X_{kt/n} - X_{(k-1)t/n}$, $k = 1,\dots,n$, and since the

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[1]In traditional gambling terminology, a martingale is the gambling system of "double-or-nothing". The fact that following this gambling system neither helps nor hurts the player's expected fortune is a particular case of a basic theorem in martingale theory.

[2]Stochastic calculus is the main topic of the companion volume to the book under review.

maximum

$$\max\{\left|X_{kt/n} - X_{(k-1)t/n}\right| : k = 1, \ldots, n\}$$

tends to 0 in probability as n tends to infinity, it follows that $X_t - X_0$ has a normal distribution, by a version of the central limit theorem due to Lévy (1934 — see [18, p. 107], [19, p. 316]), and [20, p. 2]), and to within a linear transformation of the space variables and the addition of a constant-velocity drift, X is a Brownian motion in $\mathbf{R}^{d'}$ for some $d' \leq d$. This brings us to the fact, which may be surprising to the non-expert, that the typical Markov process does not have continuous paths. For instance, if Z is the Lévy process corresponding to the innocent-looking *Cauchy semigroup* on $\mathbf{R}$,

$$\gamma_t(dx) = \frac{t\,dx}{\pi(t^2 + x^2)},$$

then it is easy to calculate that the maximum mentioned above (with X replaced by Z) does not tend to 0 in probability as n tends to infinity, so that the paths of Z cannot be continuous. In fact, it can be shown that almost all paths of the Cauchy process have a dense set of times of discontinuity. In general, the paths of a Lévy process may be taken to be right-continuous with left limits.

There is a remarkable connection between Brownian motion (X, Y) in $\mathbf{R}^2$ and the Cauchy process Z in $\mathbf{R}$. Here X and Y are independent Brownian motions in $\mathbf{R}$. Suppose X, Y, and Z all start from 0. For $y \in [0, \infty)$, let T_y be the first time that Y hits y. By a symmetry argument (Désiré André's reflection principle [1], [3, p. 153]), we have $P(T_y < t) = 2P(Y_t > y)$. Then, since Y_t has the same distribution as $t^{1/2}Y_1$ and Y_1 has a standard normal distribution, it is easy to find the probability density function for T_y explicitly. Since X and T_y are independent, it is then elementary to verify that for each fixed y, the distribution of X_{T_y} is the same as the distribution of Z_y, namely γ_y. In fact, it turns out that the process X_{T_y}, $0 \leq y < \infty$, and the process Z have the same distribution.

Now the Poisson integral formula says that if f is any bounded continuous function on $\mathbf{R}$ and if $h(x, y) = (f * \gamma_y)(x)$ for $x \in \mathbf{R}$ and $0 < y < \infty$, then h is the unique bounded harmonic function in the upper half plane with boundary values on the x-axis given by f. It follows from the previous paragraph that if Brownian motion (X, Y) in $\mathbf{R}^2$ starts at (x, y) and T is the first time that (X, Y) hits the x-axis, then the expected value of $f(X_T)$ is $h(x, y)$. This is not an accident. Kakutani showed that the solution of the Dirichlet problem in any open set D in $\mathbf{R}^d$ can be expressed in terms of Brownian motion B in $\mathbf{R}^d$ in much the same way.[3] Later, Doob [4] (see also [5, p. 651]) showed that martingale theory provides a particularly elegant proof of this striking result, including the fact that even if the boundary function is merely integrable with respect to harmonic measure (and thus perhaps highly discontinuous), the solution of the Dirichlet problem will still satisfy the given boundary condition in the sense that the limit of the solution along almost every Brownian path in D will be the value of the boundary function at the point where the path first reaches the boundary.

There are other remarkable connections between Brownian motion and potential theory besides the one with the Dirichlet problem. Here are two more. They both involve the equilibrium charge distribution of a conductor. Let us recall what

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[3] If the open set D is unbounded, then its boundary should be taken in the one-point compactification of $\mathbf{R}^d$.

this is. For each positive Borel measure μ on $\mathbf{R}^d$, where $d \geq 3$, the electrostatic *potential* of μ is the function U^μ on $\mathbf{R}^d$ defined by $U^\mu(x) = \int_{\mathbf{R}^d} G(x,y)\, d\mu(y)$, where $G(x,y) = C_d|x-y|^{2-d}$ is the *Newtonian potential kernel* for $\mathbf{R}^d$, C_d being a positive normalization constant. The *energy* of μ is $\int_{\mathbf{R}^d} U^\mu\, d\mu$, which is (twice) the amount of work that would be required to assemble the charge distribution μ from infinitesimal charges brought in from infinity. Let K be a compact set in $\mathbf{R}^d$, where $d \geq 3$. Among the positive Borel measures carried by K that have energy less than or equal to 1, there is a unique one μ_K whose total mass is as large as possible. The total mass of μ_K is denoted $C(K)$ and is called the *capacity* of K. Among positive Borel measures carried by K that have total mass $C(K)$, the measure μ_K is the unique one of smallest energy. The measure μ_K is called the *equilibrium charge distribution* for K, and its potential u_K is called the *equilibrium potential* for K. Let B be Brownian motion in $\mathbf{R}^d$. Then it turns out that u_K and μ_K can be expressed in terms of the times that B visits K. Specifically, for each $x \in \mathbf{R}^d$, if B starts from x, then $u_K(x)$ is the probability of the event that $B_t \in K$ for some $t > 0$ and $p_t(x,y)\mu_K(dy)\, dt$ is the probability of the event $\{B_L \in dy,\ L \in dt\}$ that the last exit from K after time 0 occurs from dy during dt, where $p_t(x,y)$ is the transition density for Brownian motion and $L = \sup\{t > 0 : B_t \in K\}$. This probabilistic characterization of μ_K (due in various forms to Ueno, McKean, Chung, and Getoor and Sharpe) implies that of u_K almost immediately, by integration with respect to t, because $\int_0^\infty p_t(x,y)\, dt = G(x,y)$. However, the probabilistic characterization of u_K is older and is due to Hunt. It was Hunt who saw how to generalize the connections between Brownian motion and classical potential theory, to associate a potential theory with each sufficiently regular Markov process.

One kind of "sufficiently regular" Markov process is a Feller-Dynkin process (FD process). This is a Markov process X, in a locally compact separable metrizable state space E, whose transition function $P_t(x, dy)$ acts as a strongly continuous semigroup of linear operators on the space $C_0(E)$ of continuous real-valued functions on E vanishing at infinity. The paths of an FD process can be taken to be right-continuous with left limits, like the paths of a Lévy process. Each Lévy process in $\mathbf{R}^d$ is an FD process. However, the class of FD processes is too narrow. We shall mention three points to explain why this is so. First, continuous-time Markov chains in countable state spaces have played an important role in the development of the theory of Markov processes, because they are simple enough to be studied without the use of heavy machinery but rich enough to illustrate the variety of phenomena that one should try to study. However, if X is a continuous-time Markov chain in a countably infinite discrete state space I, then X need not be an FD process; for instance, its transition function will map $c_0(I)$ into $\ell^\infty(I)$ but not necessarily into $c_0(I)$, and its paths cannot necessarily be taken to be right-continuous. Second, natural probabilistic methods of constructing new processes from old ones lead out of the class of FD processes. And third, Markov processes in infinite-dimensional state spaces are not FD processes, because their state spaces are not locally compact. To encompass natural examples such as these within the class of "sufficiently regular" Markov processes, a general procedure for re-topologizing and compactifying the state space of a Markov process was introduced by Ray and refined by Knight. Let us sketch how this may be applied in the case of chains. Let X be a continuous-time Markov chain in a countable state space I. To avoid uninteresting pathology, assume that its transition function satisfies the customary condition that $P_t(i, \{i\}) \to 1$ as $t \to 0+$ for all $i \in I$. Then in the Ray topology on I, the paths of X can be taken to be right-continuous, and they will then have left limits in the Ray-Knight compactification of I, but not necessarily in I itself. This reflects the fact that the chain may go to "infinity" in finite time and then jump back into I, as can easily be seen in simple examples.

Chapter I of the book under review is devoted mainly to a survey of the highlights of the theory of Brownian motion. Topics discussed include Brownian motion as a martingale, as a Gaussian process, and as a Markov process; Brownian local time; Skorokhod's theorem on embedding a random walk in Brownian motion; Donsker's theorem on convergence of a sequence of random walks to Brownian motion; the modulus of continuity and the quadratic variation of the sample paths of Brownian motion; the law of the iterated logarithm; windings of planar Brownian motion; self-intersections of the Brownian path; and the connections between Brownian motion and classical potential theory. The chapter concludes with material, which has been added for this edition, on Gaussian processes and on Lévy processes. Throughout Chapter I, the emphasis is on stimulation of the reader's interest by means of a survey of fascinating results. Enough proofs are given to keep the treatment from becoming superficial, but a whole book would be required to give a detailed exposition of all the results discussed in this chapter. Indeed many of these results are revisited in greater detail later in this book or in its companion volume.

Chapter II concentrates on measure theory and martingale theory. From martingale theory, it covers the usual results that are needed for the study of Markov processes and for stochastic calculus. From measure theory, it covers results that are important for probability theory but usually are not discussed in textbooks on real analysis, such as the Daniell-Kolmogorov theorem on construction of measures on infinite product spaces, the theory of weak convergence of probability measures on Polish spaces, and the construction of regular conditional probabilities (also known as disintegration of measures). Proofs are supplied for most of the results considered in this chapter.

Chapter III treats the theory of general Markov processes. Topics discussed include Feller-Dynkin processes, additive functionals, Ray processes and the Ray-Knight compactification, and applications to Martin boundary theory, time reversal (including an intuitive explanation of the probabilistic interpretation of equilibrium charge distributions), and Markov chains. Some proofs in this chapter are just sketched, or even omitted, but on the whole the exposition is reasonably detailed.

While Chapter III is an excellent introduction to Markov process, it does not give an account of the developments in this area since the original (1979) edition was published, so as the authors themselves point out, it is a little dated. It seems appropriate to mention some references from which one can learn about more recent work. In the potential theory of Markov processes, excessive functions and excessive measures are both candidates for the role that non-negative superharmonic functions play in classical potential theory, and both were discussed by Hunt. However, for a long time, excessive functions received the most attention. In the mid-1980's, the realization emerged that in fact excessive measures are the more appropriate objects of study in probabilistic potential theory. For an account of these developments, see [10]. By the way, reference [8], which is primarily concerned with an application of these results to the stopping distribution problem, also contains a summary of some of the main facts about the probabilistic potential theory of excessive measures in a form that the reader may find convenient. An area that has been studied extensively in recent years is the theory of Markov processes in spaces of measures. These have connections with population genetics models and with certain non-linear partial differential equations, among other things. See [6] and [7] for accounts of this work. For up-to-date accounts of the connections between Markov processes and Dirichlet forms, see [9] and [21].

The first edition of the book under review was a lean and lively introduction to the theory of Markov processes. This new edition retains the exuberance of the first but has gained an additional author (Rogers) and includes more topics and (especially in Chapter II) more details. In any mathematical exposition, there is always tension between portrayal of the big picture and presentation of a detailed account of the theory in a logical order. This book places greater than usual emphasis on the big picture and on presentation of key examples that have motivated the theory. This makes it a valuable addition to the literature on Markov processes.

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