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Elementary theory of L-functions and Eisenstein series, by Haruzo Hida, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993, x + 386 pp., \$69.95, ISBN 0-521-43411-4 (hardback), ISBN 0-521-43569-2 (paperback)

The romance of analysis and arithmetic is among the deepest and most enticing themes in all of mathematics. Though arithmetic, with all of its delicate subtleties, may at first seem an unlikely match for analysis with its powerful techniques, the relationship between the two has always been a vital one. In recent decades p-adic analysis—a hybrid of arithmetic and analysis—has emerged as a fascinating offspring of this union. With stunning successes in Iwasawa theory, p-adic modular forms, p-adic galois representations and their deformation theory, the field of p-adic analysis has forcefully asserted its own independence. Perhaps even more significantly, the p-adic theory is beginning to reveal some of the most intimate secrets of the mysterious relationship of analysis and arithmetic. Andrew Wiles's recent proof with Richard Taylor [12], [13] of the semistable modularity conjecture—and its corollary, Fermat's Last Theorem—is just one striking example.

Among the most successful tools for the interaction of analysis and arithmetic are the Dirichlet series, i.e. infinite series of the form

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

assumed to converge for all complex s in some right half-plane $\Re(s) > s_0$. The Dirichlet series arising in number theory are often called L-functions or zeta functions. Number theorists have never formulated a precise definition of the term "L-function", but there does seem to be fairly universal agreement that they should have the following three properties. First, L-functions should have *Euler products*, that is, factorizations of the form

$$L(s) = \prod_{\ell} P_\ell(\ell^{-s})^{-1}.$$

Here, each P_ℓ is a polynomial satisfying $P_\ell(0) = 1$, the product is over all positive prime numbers ℓ, and the infinite product converges for all s in the half-plane of convergence. Second, L-functions should have *meromorphic continuations* to the whole complex plane, and third, they should enjoy certain symmetry called *functional equations*. More precisely, there should be a non-negative integer k such that $L(s)$ and $L(k - s)$ are essentially the same function.

An L-function is a kind of generating function attached to some global arithmetic object X—say, a number field, or an elliptic curve, or a galois representation. The L-function $L(X, s)$ provides us with an analytic tool for understanding what is sometimes called the local/global principle in arithmetic. More precisely, $L(X, s)$ is constructed out of local arithmetic data attached to X, i.e. information about primes or reduction modulo powers of primes. This local data is entered into $L(X, s)$ one prime at a time via its Euler factors P_ℓ. Years of experience have conditioned

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us to expect that if the local data is entered in just the right way, then the global analytic properties of $L(X, s)$ will somehow reflect the global arithmetic properties of X. The existence of analytic continuation and functional equation is just our first hint that the L-function remembers that the local data is not independent from prime to prime but is united by a global source.

To retrieve the global content of $L(X, s)$ in more explicit form, we may examine the behavior of $L(X, s)$ at special points. We know that analytically defined quantities like the order of vanishing (or pole) and the leading coefficient of the Taylor series sometimes encode global arithmetic invariants of X. For example, the analytic class number formula expresses the residue at $s = 1$ of the Dedekind zeta function of a number field in terms of the field's fundamental global invariants: the discriminant, the class number, and the regulator. For X an elliptic curve over $\mathbf{Q}$ the conjecture of Birch and Swinnerton-Dyer predicts that the order of vanishing of $L(X, s)$ at $s = 1$ is equal to the rank of the group of rational points on X and that the leading coefficient of the Taylor series encodes the order of the Tate-Shafarevich group as well as the height regulator. Thanks to the work of many mathematicians, notably Coates and Wiles, Rubin, Gross and Zagier, Koyvagin, and Wiles [3], [5], [10], [13]) a great deal is now known about this conjecture (at least for X of rank ≤ 1).

Remarkably, many of the L-functions of arithmetic are known to assume rational values at special points (at least up to predictable transcendental factors, like a power of π or some other "period integral"). As our perspective broadens, a unified, but still highly conjectural and incomplete, picture is beginning to emerge (see e.g. Tate, Deligne, Beilinson, Bloch and Kato, among others [1], [2], [4], [7], [8], [9], [11]). Nevertheless, the problems are so daunting that a natural division of labor has taken root in which the dual problems of content and form are often investigated separately. This has given rise to a distinction between what may be called the algebraic and analytic sides of the subject. On the algebraic side one asks for the "meaning" of the special values, while on the analytic side one concentrates on their "structure". Obviously, these two points of view are inseparable, each illuminating the other, but they do have their own distinctive flavors. The book under review concentrates almost exclusively on the analytic side.

The structure of the special values (appropriately normalized to be algebraic) is usually expressed through their arithmetic properties. Indeed, we can sometimes prove that the special values of a given L-function (and its Dirichlet twists) satisfy enough congruences to guarantee the existence and uniqueness of certain p-adic analytic functions that interpolate essentially the same values at the special points as the complex L-function. Whenever this happens (which is quite frequently) we find ourselves in the happy circumstance of having not just the original complex L-function $L(X, s)$, $s \in \mathbf{C}$, but also an infinite array of p-adic L-functions $L_p(X, s)$, $s \in \mathbf{Z}_p$, one for each prime p. This array of L-functions is held together by its common skeleton of special values and offers us the opportunity to study these special values using either complex or p-adic analytic means. Even more remarkable is the apparent fact that the p-adic analytic properties of $L_p(X, s)$ mirror those of the complex L-function. For example, in most cases (but not all; see [6] for a fascinating example) the order of vanishing of the p-adic L-function at any of the special points is the same as the order of vanishing of the complex L-function at the same point. The introduction of p-adic analysis gives us a major advantage, since

p-adic analysis is much closer to arithmetic than complex analysis and is therefore more readily related to the global arithmetic problems that interest us.

There is another (unexpected and quite beautiful) advantage of working with p-adic analysis alongside complex analysis. This comes from the possibility of p-adically deforming global arithmetic. A family of global arithmetic objects may appear discrete and disjointed in the complex optic, while the p-adic optic may reveal that certain features of this family fit together into coherent p-adic analytic families. As one might imagine, p-adic L-functions are particularly well-suited for expressing this analyticity. As we move X in an appropriate family, the associated p-adic L-function may move in a corresponding p-adic analytic family, giving rise to a p-adic L-function of several variables. Haruzo Hida, the author of the book under review, has been a major proponent of this point of view since about 1980, when he introduced his groundbreaking work on families of modular forms. It is significant to note that Hida's ideas later motivated Barry Mazur to introduce the theory of p-adic deformations of galois representations, which eventually played a central role in Wiles's proof of Fermat's Last Theorem.

The book under review is a tour of the elementary analytic theory of L-functions, especially those L-functions coming from algebraic number fields and from modular forms. Its emphasis on the p-adic aspects of the theory distinguishes it from any other text previously written on the subject.

The book begins with a concise review in Chapter 1 of algebraic number theory and the basics of p-adic analysis. The analytic theory of Hecke L-functions is treated in Chapters 2 and 8. Chapter 2 establishes the basic analytic properties of Hecke L-functions, namely, that they have meromorphic continuations and that their special values at negative integers are algebraic. Before developing the general results, special results for the Riemann zeta function, Dirichlet zeta functions, real quadratic fields and imaginary quadratic fields are proven. The unifying tool is Shintani's zeta function, which, in preparation for future chapters, is also used to prove analytic continuation of Eisenstein series. Later, in Chapter 8, the adelic approach is introduced and used to prove the functional equations for Hecke L-functions.

The p-adic L-functions attached to Dirichlet L-functions and Hecke L-functions for totally real fields are constructed in Chapter 3. The author follows Katz's approach, interpreting the methods of Cassou-Nogués and Barski in terms of formal groups. Chapter 4 gives still another account of these results over $\mathbf{Q}$ based on what the author calls the "modular symbols method".

The analytic theory of modular forms is developed in Chapters 5, 6, and 9. Chapter 5 introduces the space of elliptic modular forms and proves some of their basic properties. For example, Shimura's theorem on the rationality of the space of modular forms is proven in depth. The algebra of Hecke operators is introduced, and its significance for Euler products of standard L-functions attached to modular forms is explained. Also, the Rankin product L-functions $D(s, f, g)$ are described briefly. In Chapter 6, the cohomological point of view on modular forms is introduced and the Eichler-Shimura isomorphism theorem is proved. This is used to prove Shimura's fundamental result that special values of L-functions attached to Hecke eigenforms are essentially algebraic. Mazur's construction of p-adic L-functions is reviewed quickly at the end of Chapter 6. The properties of analytic Eisenstein series are developed in Chapter 9, where adelic methods are used to compute Fourier expansions of Eisenstein series, analytic continuation and functional equations for the

Eisenstein series. These results are then applied to establish analytic continuation and functional equation for Rankin product L-functions.

The heart of the book is found in Chapters 7 and 10. Here the author describes his theory of p-adic analytic familes of ordinary modular forms (the so-called ordinary Λ-adic modular forms) and applies the theory to construct three-variable p-adic Rankin product L-functions. In particular Chapter 7 gives a quite readable account of Wiles's approach to Hida's theory. Especially nice is the account of Wiles's notion of pseudo-representations and the application to constructing p-adic analytic families of galois representations attached to ordinary Λ-adic modular forms.

On the whole, the book concentrates on the analytic side of the subject, but there are a few places where the algebraic side is discussed, notably the proof of the analytic class number formula in Chapter 1 and the discussion of Λ-adic galois representations in Chapter 7. The final "Concluding Remarks" section of the book also gives a brief overview with detailed references to not only the algebraic theory, but also the closely related geometric and automorphic theories.

The author has made an important contribution to the literature by gathering under one cover a significant collection of ideas which earlier had only been available in research journals. One of the nicest features of the book is that it treats the author's theory of ordinary Λ-adic modular forms in a self-contained fashion. This is a subject that has grown in importance over the years, and its appearance in book form is quite timely. Researchers will certainly appreciate the enhanced accessibility of these ideas, as well as the valuable insights the book gives into Professor Hida's personal view of the subject.

At times, however, the author may have tried too hard to treat too many themes from too many points of view. This leads to a dense style that sometimes obscures the main lines of the book. For example, the reviewer wonders about the author's decision to introduce group schemes and formal groups in an elementary chapter on p-adic L-functions. Although beginning graduate students will probably have difficulty using *Elementary theory of L-functions and Eisenstein series* as a textbook, the topics are certainly well chosen and worth mastering. Anyone who studies Hida's book carefully will appreciate the variety of approaches and methods and may then choose from among them according to his or her own personal tastes.

The book would have benefitted from more careful proofreading. The index is difficult to use and often does not refer the reader to what one needs to find. For example, a general definition of the term Hecke character, a major theme of the book, cannot be found by checking the index.

In conclusion, this is a comprehensive and important book—one that deserves to be studied carefully by any serious student of L-functions and modular forms.

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