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Diagram geometries, by Antonio Pasini, Oxford Sciences Publications, Oxford University Press, 1994, vii+488 pp., $95.00, ISBN 0 19 8534 97 3

This book is an invitation to enter the world of diagram geometry and to play the following game: investigate incidence geometries starting from some knowledge of their local 2-dimensional structure.

Incidence geometry can be found almost everywhere in mathematics, since it is related to the most basic and elementary mathematical structures. The idea progressively emerged from the work of nineteenth-century geometers like Pasch and von Staudt, but it is usually credited to Hilbert. In his masterpiece *Die Grundlagen der Geometrie* (1899) he set up a system of axioms with different levels concerning undefined concepts like point, line, plane, lie on, betweenness, congruence, etc., pointing out that points, lines, and planes could be replaced by chairs, tables, and beer mugs, or other objects—their true nature and our mental representation of them being irrelevant.

The first level consists of connection axioms, e.g., "for each two points there is one line which lies on these two points". Other levels concern betweenness, congruence, parallelism, and continuity. If we call *incidence* the symmetrization of the "lie on" relation, the first level of this system of axioms defines a class of incidence geometries.

Incidence geometry generalizes not only projective and affine geometry but also the geometries induced on such spaces when some additional structure (e.g., a quadratic form) is given. It also encapsulates much of the discrete mathematics developed in this century for purposes ranging from the very practical to the very theoretical. Let us mention, for example, *designs*, created to design plant experiments and first studied as a part of applied statistics; *coding theory*, created for the detection and correction of errors of data transmission; *matroids*, having applications in the study of rigid structures in architecture; and classical *finite geometries* as they relate to linear representations needed for the investigation of finite groups. Such discrete structures are of interest not only for mathematicians, statisticians and computer scientists but also for physicists and engineers.

Incidence geometry fits into two algebraic frameworks. One of them is *linear algebra*, through the notion of division ring or through the use of incidence or adjacency matrices; the other one is *group theory*, starting with Klein's *Erlanger Programm*, presenting the study of geometries as that of certain permutation groups (namely, their automorphism groups). Nevertheless, neither of the two algebraic languages superseded the geometric viewpoint. Indeed some geometric intuition enriches both linear algebra and group theory. For example, compare the simplicity of the "three axioms" defining a projective plane to the "twenty axioms" used to define a division ring! But above all, geometry contributed to group theory in the study of Lie-Chevalley groups, Coxeter groups and sporadic simple groups...

This was the beginning of an exceptionally fruitful interplay between geometry and group theory, often supported by highly synthetic drawings named *diagrams*. The investigation of *complex simple Lie groups* amounts to that of *Lie algebras*,

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characterized by their *root systems*, represented in turn by the famous *Dynkin diagrams* (1948).

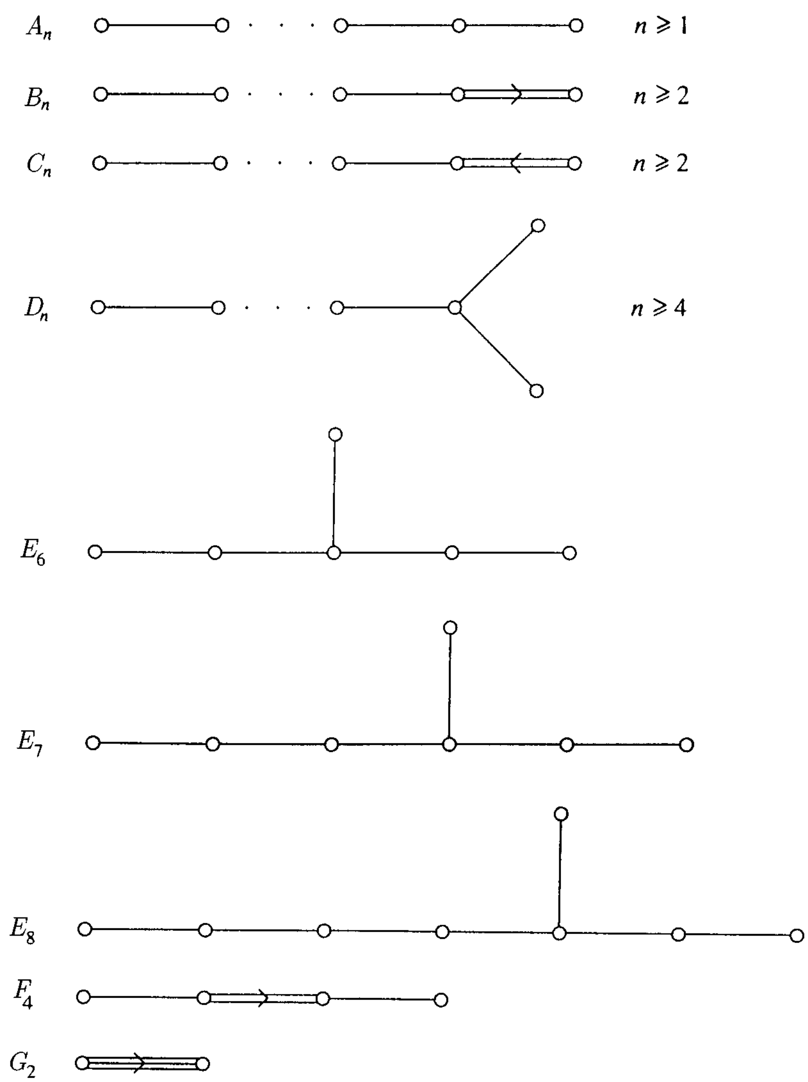
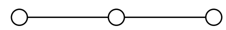


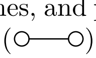
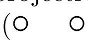
FIGURE 1

While the classical groups (corresponding to A_n, B_n, C_n and D_n) could be interpreted as the automorphism groups of certain geometries, such a geometrical interpretation was missing for the exceptional ones (related to E_6, E_7, E_8, F_4 and G_2). Trying to fill this gap, Tits built a bridge between groups and geometries. It was known that the vertices of the Dynkin diagrams correspond to the conjugacy classes of maximal parabolic subgroups. Tits then associated a geometry Γ to the Dynkin diagram M of any complex semisimple group G, taking as elements of Γ the maximal parabolic subgroups (partitioned into "types": the conjugacy classes) and defining any two such subgroups to be incident if their intersection contains a maximal connected solvable subgroup. In this way, Klein's identification "geometry

↔ permutation group" was refined to a new identification "incidence geometry ↔ group together with distinguished classes of subgroups and some incidence rule". As a result, the points lost their prominent role as the atoms of the geometry. But above all, Tits observed the following two critical facts.

First, the neighborhood of any element x of Γ (called the *residue* of x) is the geometry associated with the diagram of M by *deleting the vertex* of M corresponding to the type of x. For example, in a 3-dimensional projective geometry associated with the Dynkin diagram



whose three vertices correspond respectively to points, lines, and planes, all residues of points and all residues of planes are projective planes (), while all residues of lines are so-called generalized digons (), i.e., geometries of rank 2 in which every element of one type is incident with every element of the other type.

Second, the knowledge of the geometries associated with the Dynkin subdiagrams of rank 2 (i.e., with two nodes) of M almost suffices to characterize Γ. This led Tits to create diagram geometry, "a device to define 'complicated' geometries by means of simpler ones, like building blocks" [5]. Having noticed that the geometries corresponding to rank 2 subdiagrams of the Dynkin diagrams were simple, well known, and could be defined over any field, he was able to define new geometries over any field k and hence new groups. In the meantime, Chevalley published his elegant construction of algebraic semisimple groups over arbitrary fields, so that Tits's "new groups" were not new any more. But the geometric part of his work remains, where he generalized Dynkin diagrams. He first defined rank 2 geometries that he called *generalized n-gons*, because their simplest incarnations are ordinary n-gons. He showed that the class of *Coxeter diagrams*, i.e., diagrams made of strokes symbolizing n-gons, together with additional axioms, represents geometries from which the main properties of Chevalley groups could be read. But Tits went on and created buildings, a more appropriate tool for his quest.

Wandering about in the wide world of geometrical structures and looking for geometries describing sporadic groups, Buekenhout noticed the important role of some special, trivial rank 2 geometries, which he called *circle geometries*. Although the addition of strokes symbolizing circle geometries suffices to associate a diagram and a geometry to every sporadic simple group, he weakened the axioms of Tits as far as he could to get a very general notion of *diagram geometries*, sometimes called *Buekenhout geometries*, investigated in the book under review. Let us now turn to its content.

Incidence geometries are usually defined by their incidence graph, whose vertices are the *elements* of the geometry and whose adjacency is the *incidence* relation. Since there are several inequivalent definitions, the author chose a simple and efficient one, by induction on the rank. A *rank 1 incidence geometry* is just a graph with at least two vertices and no edge. A *rank n incidence geometry* Γ is a connected n-partite graph such that the neighborhood of any vertex is a rank $n-1$ geometry whose $n-1$ classes are subsets of the n classes of Γ. Two elements have the same *type* if and only if they are in the same class. A *flag* is a set of pairwise incident elements, and its *residue* is the geometry induced on the intersection of the neighborhoods of the elements of the flag.

For example, if Γ is a rank n projective geometry, then the elements of type i $(0 \le i \le n-1)$ are the i-dimensional subspaces of Γ. Any nest of $n-2$ subspaces is

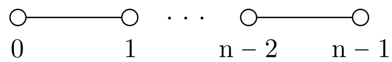


Figure 2

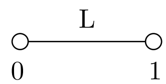


Figure 3

Figure 4

a flag F and has a residue of rank 2, which is a projective plane if the dimensions of the subspaces in F range from 0 to $n-3$ or from 2 to $n-1$. However, the residue of F is a generalized digon (i.e., a complete bipartite graph) in the other cases. The diagram usually associated with Γ is shown in Figure 2, where each node stands for a class of Γ; here, the node with subscript i represents the class of all i-dimensional subspaces of Γ. A single stroke joining two nodes i and j means that any residue of type $\{i, j\}$ is a projective plane. The absence of a stroke connecting the nodes i and j means that any residue of type $\{i, j\}$ is a generalized digon.

It is remarkable that this diagram characterizes the class of n-dimensional projective geometries but not Γ itself. Indeed, we could be more precise in our description of Γ by giving explicitly the orders of all the projective planes, but we could also be less precise by replacing the strokes denoting the class of projective planes by strokes denoting a wider class, for example, the class of linear spaces, represented by Figure 3. (These last are the geometries consisting of points (elements of type 0) and lines (elements of type 1) such that any two points are incident with precisely one line and any line contains at least two points.) Then Γ admits also Figure 4 as a diagram. But any simple matroid of rank $n+1$ admits the same diagram.

A *diagram geometry* Γ is an incidence geometry in which, for any pair (i, j) of types, either all or none of the residues consisting of elements of these types are generalized digons. Then a *diagram* can be associated with this geometry by joining any two types i and j by a stroke labelled by any class of rank 2 geometries containing all residues of Γ on i and j, the strokes symbolizing generalized digons being omitted.

This book helps us to play the game "*given a diagram, find all geometries admitting this diagram*". In the first three chapters, a wealth of examples are produced before and after introducing the basic notions; the elementary theory is developed in the next three chapters, preparing for Chapter 7, which contains classification theorems of geometries with prescribed diagrams (e.g., L_n, C_n, D_n, and variations) and characterizations of matroids, polar spaces, D_n buildings, etc.

Chapters 8 to 12 introduce more sophisticated notions and tools involving automorphism groups, quotients, and universal covers. These are used in the last three chapters, which go further in the classification game, focusing on Coxeter diagrams of spherical type and on C_n-like diagrams.

This book invites readers to work with diagram geometries rather than to collect all results obtained so far. Such a collection, updated in 1992, is available in Chapter 22 (by Buekenhout and Pasini) of the *Handbook of incidence geometry* [3]. The book under review does not investigate the "point-line" approach of geometries of dimension ≥ 3, leaving this topic to Buekenhout and Cohen's book [4]. It does not specialize in buildings either, whose theory is fully developed in the books of Tits [6], Brown [1], and Ronan [5]. However, it is an excellent and truly readable book for all those who want to get acquainted with diagram geometry or to develop their skills in working with such geometries.

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Anne Delandtsheer
Universite Libre de Bruxelles
E-mail address: adelandt@ulb.ac.be