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Limit theorems of probability theory: Sequences of independent random variables, by Valentin V. Petrov, Oxford Studies in Probability, vol. 4, Oxford Science Publications, Clarendon Press, Oxford, 1995, ix + 292 pp., \$90.00, ISBN 0-19-853499-X

The simple notion of statistical independence lies at the core of much that is important in probability theory. First there was the classical *central limit theorem* (CLT) of De Moivre and Laplace which, in its final form due to Paul Lévy, says that the sum S_n of n independent and identically distributed (i.i.d.) random variables X_j $(1 \leq j \leq n)$ having finite second moments is asymptotically Gaussian, or *normal*, as $n \to \infty$. Then came the *stable laws*. If one did not assume finiteness of second moments, what could be the possible (nondegenerate) limit laws of $a_n + b_n S_n$ under scaling sequences a_n and $b_n > 0$? The limit need not exist. But if it does, say Q, then it should clearly be "stable": whatever be $n \geq 1$, if X_j are i.i.d. with distribution Q, then there exist a_n and $b_n > 0$ such that $a_n + b_n S_n$ has distribution Q. The characteristic function, or the Fourier transform, of a symmetrized stable law is of the form $\exp\{-b|t|^\alpha\}$, where $b > 0$, and α is the "index" of the stable law, $0 < \alpha \leq 2$. The case $\alpha = 2$ corresponds to the normal law. The *infinitely divisible laws* soon followed as the possible limit laws of partial sums of independent random variables not necessarily identically distributed, but individually small ("asymptotically negligible"): Q is "infinitely divisible" if for each n one may express Q as the n-fold convolution Q_n^{*n} of some probability measure Q_n. The stable laws constitute only a small subset of the set of all infinitely divisible laws. The *Lévy-Khinchine representation* of the characteristic function of an infinitely divisible law displays it as the law of a limiting sum of independent (compound) Poissonian jumps of different sizes, with a possible Gaussian component. The theory outlined above is a cornerstone of classical probability. Although it is an unenviable task to compete with the beautiful account of this theory given in Gnedenko and Kolmogorov [13], Chapters 3 and 4 of Petrov's book give a good streamlined presentation. A wealth of information on this topic, as well as various approaches to it, may be found in Feller [11].

On the topic of convergence in law, the book also contains a substantial chapter (Chapter 5) on rates of convergence to the Gaussian law in the central limit theorem. One of the most important results here is the *Berry-Esseen bound*, which asserts that for zero-mean random variables X_j, with variance $\sigma^2 > 0$ and a finite absolute third moment ρ_3, the difference between the distribution function $F_n(x)$ of $S_n/n^{1/2}$ and the distribution function $\Phi(x)$ of the Gaussian law, with the same mean and variance as X_j, is bounded above by $c\ \rho_3/\ (\sigma^3 n^{1/2})$ for some absolute constant c. There is an appropriate extension of this bound for independent, but non-identically distributed, summands. Although the Berry-Esseen bound cannot be improved upon in general, Cramér [5] had earlier obtained asymptotic expansions of $F_n(x)$ in powers of $n^{-1/2}$ assuming finiteness of additional moments and the so-called *Cramér's condition (C)*: the characteristic function of X_1 is bounded away from one in magnitude at infinity. There are various other refinements of the CLT. For

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example, if $x >> (\text{logn})^{1/2}$, then the Berry-Esseen bound contains no information, and for this case error bounds depending on x are available. In particular, if the moment generating function $t \to E(e^{tX})$ is finite for all t in some neighborhood of $t = 0$, then for $x = o(n^{1/6})$ Cramér showed that $(1 - F_n(x))/(1 - \Phi(x)) \to 1$, as $n \to \infty$. In the present book the author presents his own extension of Cramér's last cited result to the case $x = o(n^{1/2})$. A notable omission in this chapter is Esseen's asymptotic expansion for the case of lattice summands. Esseen's classic article [10] still remains a source of valuable information on the topic of Chapter 5, including an application to the lattice point problem of analytic number theory. See Bentkus and Götze [2] for recent progress on this classical theory of Landau, using probabilistic methods.

Apart from convergence in law of (scaled) sums S_n of independent random variables, the other major topic in Petrov's book is almost sure (a.s.) convergence of properly normalized sums. For i.i.d. summands, the classical *strong law of large numbers* (SLLN) was proved by Kolmogorov a few years before Birkhoff obtained his ergodic theorem. Two decades earlier, Borel [4] had proved a special case of the SLLN: the set of *normal numbers* in the unit interval has full Lebesgue measure. Recall that a "normal number to base p" is one whose p-adic expansion contains the same limiting proportions of the digits $0, 1, \ldots, p-1$, namely, $1/p$. A "normal number" is one which is normal to every base $p > 1$. Borel's result was significantly refined by Hausdorff (in 1913) and Hardy and Littlewood (in 1914). But the most precise and the deepest refinement for dyadic normal numbers was provided by Khinchine [16]: Let p_n be the proportion of 0's among the first n digits of the dyadic expansion of a number in the unit interval; then $\limsup\, (p_n - 1/2)\{n/(2 \log \text{logn})\}^{1/2} = 1$ a.s. The corresponding $\liminf$ is -1 a.s., by symmetry. Later Kolmogorov [17] obtained a far-reaching extension of this law of iterated logarithm applicable to sums of a broad class of independent, but not necessarily i.i.d., sequences. In particular, Khinchine's refinement extends to normal numbers to all bases p. One culmination of these results is the Hartman-Wintner *law of iterated logarithm* (LIL) for i.i.d. sequences: If $EX_1{}^2 < \infty$, then $\limsup\, (S_n - nEX_1)/\{2n \log \text{logn}\}^{1/2} = \sigma$ a.s., where σ is the standard deviation of X_1. By symmetry, the $\liminf$ is $-\sigma$ a.s. Indeed, a powerful result of Strassen [21] implies that the set of all limit points of this normalized sequence of partial sums is the closed interval $[-\sigma, \sigma]$, a.s. Chapters 6 and 7 of Petrov's book present these and other related results, including some due to the author.

In addition to the contents described above, Chapter 2 in the book has a number of useful inequalities for sums of independent random variables. Some of these neatly derived inequalities, such as Rosenthal's inequality for moments of S_n and concentration inequalities of Lévy, are important but not found in graduate texts in probability.

For the potential reader of the present monograph it is helpful to know that there are (1) fairly complete extensions of the main results to more general structures and (2) closely related important topics of current interest, as described below.

The theory of limit laws for normalized partial sums of independent random variables presented in this book extends to higher-dimensional euclidean spaces, second countable locally compact abelian groups, and infinite-dimensional separable Hilbert spaces (Parthasarathy [19]). In the case of locally compact abelian groups, the sum "+" is the group operation, and characters ξ take the place of the exponential functions $\xi : x \rightarrow e^{i\xi x}$. Significant extensions have also been made to infinite-dimensional Banach spaces (Araujo and Giné [1]).

Among important closely related topics not included in the monograph are the functional versions of the central limit theorem and the law of iterated logarithm. For i.i.d. summands with mean zero and finite positive variance σ^2, the functional central limit theorem of Donsker [6] asserts that the sequence of $C[0,1]$-valued random maps X_n, defined by $X_n(t) := S_m/n^{1/2}$ for $t = m/n$ $(m = 0, 1, 2, \ldots, n)$, $S_0 = 0$, with extension by linear interpolation to [0,1], converges in law to the Wiener measure. This really says that at appropriate large scales of time and space the random walk $\{S_n : \; n = 0, 1, 2, \ldots\}$ looks like a Brownian motion—a process in continuous time with continuous sample paths, whose increments over disjoint intervals are independent and Gaussian. The independence of increments of the limiting process is immediate from that of the random walk, and Gaussianness follows from the CLT. That such a process exists with continuous sample paths should perhaps be expected from Einstein's 1905 article on the Brownian movement (see [9]), but it is a famous result of Wiener [22]. The other functional law, that of the iterated logarithm, is due to Strassen [21]. It says that the set of limit points of the sequence of functions $Y_n(t) := X_n(t)/(2 \log \text{logn})^{1/2}$, $0 \leq t \leq 1$, is given by $K = \{f \epsilon C[0,1] : f$ absolutely continuous, $f(0) = 0$, $\int_0^1 (f'(t))^2 \; dt \leq \sigma^2\}$, a.s.

The topic of infinitely divisible laws directly leads to Lévy's theory of processes with independent increments, Brownian motion being the only such process with continuous sample paths. Just as Brownian motion is built of independent homogeneous Gaussian increments, each infinitely divisible law is the germ of a homogeneous process with independent increments. Lévy and Itô found a representation of the sample paths of these processes as made of independent Poissonian jumps of different sizes, with a possible Brownian component (Lévy [18], Doob [7], Itô [14]). If one wishes to travel a little further, starting from Brownian motion another step leads to Markov processes in continuous time which are locally like a Brownian motion. These are nonanticipative solutions of stochastic differential equations, the stochastic integration being with respect to Brownian increments. Averages of functions, or functionals, over the path space of these processes provide solutions of second-order parabolic and elliptic partial differential equations (Dynkin [8], Itô [14], Friedman [12], Karatzas and Shreve [15]).

As a final comment, it may be pointed out that the main applications of refinements of the central limit theorem described in Chapter 5 of the book have been in the large sample theory of statistics. But even for the simplest of such applications the one-dimensional theory presented is inadequate, and one needs error bounds and asymptotic expansions for the multi-dimensional CLT. Of course, it would have taken many more pages to provide an adequate account of this extension. For an account of multi-dimensional results, see [3]. Also, a more comprehensive presentation of refinements of the one-dimensional CLT than given in Chapter 5 may be found in Petrov's earlier popular monograph [20].

To summarize, in seven chapters the book under review gives a well-organized and readable account of two important topics in probability: asymptotic distributions of sums of independent random variables and their refinements (Chapters 3, 4, 5) and almost sure convergence properties of these sums (Chapters 6, 7). The first two chapters develop the necessary background. Chapter 1 introduces some basic notions in probability and provides essential facts on characteristic functions and

Lévy's concentration. Chapter 2 is on inequalities relating to sums of independent random variables. Among the distinguishing features of the book are "addenda" at the end of each chapter giving additional, and more specialized, results due to various authors. There are few misprints. Only one minor comment on the format: it would have been better if definitions were italicized or otherwise clearly distinguished in the body of the text. The monograph is a fine addition to the literature on sums of independent random variables. The only prerequisite for reading it is a one-semester graduate course in probability.

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