BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, January 1997
S 0273-0979(97)00702-7

Lie group actions in complex analysis, by Dmitri N. Akheizer, Aspects of Mathematics, vol. E27, Friedr. Vieweg, Braunschweig and Wiesbaden, 1995, vii + 201 pp., $49.00, ISBN 3-528-06420-X

Lie transformation groups have been an important object of study in mathematics for more than a hundred years. The "modern era" began with work in the forties and fifties, when many results were found in both the differentiable and algebraic categories. In recent years there has been considerable progress concerning the holomorphic actions of Lie groups on complex spaces. Initially such work focused on compact spaces, but there are now several techniques which help make non-compact spaces accessible. Some of the powerful methods which are applied here come from complex analysis in one and several variables, algebraic geometry, and Lie theory, including the theory of Lie groups and Lie algebras and representation theory.

For the purposes of this review we will consider complex spaces, i.e., spaces which are locally modeled on the common zero sets in $\mathbb{C}^n$ of finitely many holomorphic functions (functions having convergent power series in some neighborhood of every point). Let $\mathrm{Aut}(X)$ denote the group of biholomorphic maps of X onto itself. If X has complex dimension one (X is a Riemann surface), then $\mathrm{Aut}(X)$ is always a Lie group. In higher dimensions this is not true in general. However, if X is a compact complex space or if X is a bounded domain in some $\mathbb{C}^n$, then $\mathrm{Aut}(X)$ is a Lie group. In general, one has an "action" of a Lie group G on a complex space X if there is a continuous homomorphism of G into $\mathrm{Aut}(X)$. This means that many different kinds of examples of Lie group actions on complex spaces exist.

Given an action of G on the space X, each $g \in G$ is a biholomorphic automorphism of X, and we will denote the image of $x \in X$ by $g \cdot x$. The set $G \cdot x = \{g \cdot x | g \in G\}$ is called the orbit of the point $x \in X$. Now if a complex Lie group G acts transitively on a complex manifold X, i.e., if for any $x \in X$ one has $G \cdot x = X$, then it is well known that X is biholomorphic to the coset space G/H with its natural G-invariant complex structure, where H is a closed, complex subgroup of G. Such a manifold is called a homogeneous complex manifold. If a complex space X admits a holomorphic G-action which has an open G-orbit, necessarily of the form G/H, then X is called almost homogeneous.

We now describe some of the basic tools which are used in the classification of homogeneous and almost homogeneous spaces. The maximal connected normal solvable subgroup of a Lie group G is called its radical. The radical R is closed and is complex if G is complex. A Lie group is called semisimple if its radical is trivial. Every Lie group admits a Levi-Malcev decomposition $G = S \cdot R$, where S is a maximal semisimple subgroup of G, unique up to conjugation. Basically one would like to have a decomposition of a homogeneous space in terms of the orbits of R (resp. of S), because there are many special results known for homogeneous spaces of solvable (resp. semisimple) groups due to the existence of a finite composition series for these (resp. due to the theory of algebraic groups). Since R is normal in G, one obvious idea would be to consider the subgroup $R \cdot H$ and the fibration

1991 *Mathematics Subject Classification.* Primary 32M05, 32M10, 32M12.

$G/H \to G/R{\cdot}H$. However, the subgroup $R{\cdot}H$ is not closed in G, in general. Similar remarks also apply to the commutator subgroup G' of G. One important ingredient in the study of homogeneous complex manifolds has been the normalizer fibration. Its base is "essentially given" by a linear group, and by a theorem of Chevalley commutator fibrations do exist for linear groups. The fibers of the commutator fibration are the orbits of an algebraic group, and so they admit a radical fibration. As well, the fiber of the normalizer fibration has the form L/Γ, where Γ is a discrete subgroup of a complex Lie group L. By a theorem of Auslander [2] one knows that the connected component of the closure of Γ times the radical of L is a solvable group. This has provided a method which compensates for the fact that the radical orbits are not closed; see, e.g., [5]. We do not describe such methods in detail, as such a description would take us too far afield. The interested reader is referred to Chapter 3 in Akhiezer's book for more details about the normalizer fibration and to Chapter II in [8] for general methods.

In the above we are given G/H, and we found a closed, complex subgroup I of G containing H. Thus we could consider the "homogeneous fibration" $G/H \to G/I$ given by the map of cosets $gH \mapsto gI$. But one would also like to find subgroups which somehow reflect the analytic structure or the geometry of the homogeneous space G/H. We now outline how this works for the reduction with respect to the algebra of holomorphic functions on G/H.

To begin, suppose X is an arbitrary connected complex space. For $x_1, x_2 \in X$ define $x_1 \sim x_2$ precisely if $f(x_1) = f(x_2)$ for every holomorphic function f on X. This defines an equivalence relation on X, and we denote the equivalence class of x by $[x]$ and let $\pi : X \to X/\sim$ be the natural map. For X holomorphically convex this is the Remmert reduction: the space $X/\sim$ is Stein, and π is a proper holomorphic map of X onto $X/\sim$. But there are examples of complex manifolds X such that $X/\sim$ need not be locally compact. However, this procedure does work for homogeneous spaces for the following reason. Suppose there were some $g \in G$ with $g \cdot [eH] \not\subset [gH]$. Then there would exist points $x_1, x_2 \in [eH]$ with $gx_1 \not\sim gx_2$. Hence there would be a holomorphic function f on G/H with $(f \circ g)(x_1) = f(gx_1) \neq f(gx_2) = (f \circ g)(x_2)$. Since $f \circ g$ is also a holomorphic function on G/H, this would contradict the fact that $x_1, x_2 \in [eH]$. Hence

$$g \cdot [eH] = [gH] \tag{1}$$

for every $g \in G$. (Note that every $g \in G$ maps an equivalence class onto an equivalence class, since g^{-1} exists and the action of g^{-1} "undoes whatever the action of g did.") This implies $g \cdot [x] = [g \cdot x]$ for every $g \in G$ and $x \in G/H$. It is now an easy exercise, which is left to the reader, to check that

$$I := \{g \in G | [gH] = [eH]\} \tag{2}$$

is a subgroup of G containing H and that $I = p^{-1}([eH])$, where $p : G \to G/H$ is the holomorphic map given by $g \mapsto gH$. Because $\sim$ is defined in terms of the level sets of the algebra of holomorphic functions on the complex manifold G/H, its equivalence classes are closed, complex submanifolds of G/H. Hence I is a closed, complex subgroup of G! Thus G/I admits a natural complex structure, and the homogeneous fibration $\pi : G/H \to G/I$ is called the holomorphic reduction of G/H. By construction, G/I is holomorphically separable and every holomorphic function on G/H is the composition of π with a holomorphic function on G/I.

Before describing any results in the complex analytic category, we would first like to present a contrasting example from the algebraic category. Suppose G is a reductive algebraic group and H is an algebraic subgroup of G such that a Borel subgroup B of G has an open orbit in G/H. Then H is called spherical. If X is a normal irreducible almost homogeneous G-variety with the open orbit isomorphic to G/H, where H is a spherical subgroup of G, then X is called a spherical variety. The geometry of spherical varieties has many aspects in common with the geometry of flag varieties, symmetric spaces, and toroidal embeddings. The group G has finitely many orbits on X; these are themselves spherical, and there are notions of complexity and rank which aid in understanding the structure. Methods here strongly rely on the theory of algebraic groups. We do not describe any of these in detail, but refer the reader to [1, 4, 11].

Now suppose G/H is a complex homogeneous space; i.e., G is a complex analytic Lie group and H is a closed, complex subgroup. One can apply some of the fibration methods outlined above with the aim of reducing to other spaces, perhaps where one or more is even algebraic and maybe even spherical. We mention just a few examples. Homogeneous complex manifolds have been classified in dimensions two and three—e.g., see [12]—basically by using the fibration methods we have described along with various other tools. Next we suppose that X is an almost homogeneous space which is normal and has an isolated point in the complement of the open orbit. Then X is either an affine or projective homogeneous cone with its vertex removed; see [9]. The proof shows that X admits a fibration with 1-dimensional fiber over a flag manifold. Now suppose the homogeneous complex manifold G/H has more than one end in the sense of Freudenthal. Then conditions are known under which G/H can have at most two ends, e.g., if G/H has nonconstant holomorphic functions. If $X = G/H$ with $\mathcal{O}(X) \neq \mathbb{C}$ has two ends, then the fiber of the holomorphic reduction of X is connected and compact and the base fibers as a $\mathbb{C}^*$-bundle over a flag manifold; i.e., this base is an affine homogeneous cone with its vertex removed; see [5]. If G/H has three or more ends, then these ends "essentially come from" a space of the form S/Γ, where S is a semisimple complex Lie group and Γ is a Zariski dense discrete subgroup of S; see [6]. The proof here depends on the fact that the base of the normalizer fibration of G/H must be compact and the fiber "inherits the ends". Using the theorem of Auslander cited above, one sees that the radical orbits in the fiber of the normalizer fibration are compact. Thus one can mod out by the radical, and the ends come from a space of the form S/Γ. Explicit examples of homogeneous complex manifolds with more than two ends can be constructed in the following way. Consider the Bianchi group $\Gamma := SL(2, \mathcal{R}_d)$, where $\mathcal{R}_d$ is the ring of the integers in the imaginary quadratic number field $\mathbb{Q}[\sqrt{-d}]$, with d a square free positive integer, and set $S := SL(2, \mathbb{C})$. L. Bianchi [3] showed that $SL(2, \mathbb{C})/SL(2, \mathcal{R}_d)$ has three ends for $d = 7$ and four for $d = 3$.

The first chapter of Akhiezer's book deals with some basics about Lie groups and their actions on complex spaces. This includes some results on one-parameter transformation groups, vector fields on complex spaces, and the extension problem of local G-actions to global ones on a complex space. Automorphism groups are the subject of the second chapter. In particular, $\mathrm{Aut}(X)$ is shown to be a topological group; and if X is a compact complex space or a bounded domain, then $\mathrm{Aut}(X)$ is proved to be a Lie group. Some tools which are used in the compact case come from the theory of topological transformation groups and are not proved in Akhiezer's book. However, the reduction to the topological setting is proved by means of the identity theorem for compact groups of holomorphic transformations with a fixed point. This is derived from the local linearization theorem. The proof for bounded domains is connected with the fact that the action is proper, and thus proper actions are introduced and studied. This chapter also contains some explicit calculations of automorphism groups, e.g., of the ball and the polydisk, along with a characterization of the ball as the strictly pseudoconvex bounded domain whose automorphism group is noncompact.

Chapter 3 is devoted to compact homogeneous complex manifolds. The main tool here is the normalizer fibration, which Akhiezer calls the Tits fibration, as its importance in this subject was underlined by Tits. The base of the Tits fibration is a flag manifold of a semisimplex complex Lie group, and its fiber is a compact complex parallelizable manifold. Akhiezer develops rather detailed descriptions of flag manifolds, including some work with root systems, a proof that a flag manifold admits an equivariant embedding into some complex projective space, and a discussion of the automorphism groups of flag manifolds. Parallelizable manifolds are also considered. As well, there is an exposition about the role of the fundamental group, including some results of the author when $\pi_1(G/H)$ is nilpotent (then $G = S \times R$, with R nilpotent) and when $\pi_1(G/H)$ is solvable. The fourth chapter is concerned with homogeneous vector bundles over homogeneous complex manifolds G/H. There are induced representations of the group G on the cohomology groups of the base G/H with coefficients in certain sheaves, and for G/H compact these representations are shown to be holomorphic. A proof of Bott's theorem, due to M. Demazure, along with some applications is given. These include computation of the cohomology of the tangent sheaf over flag manifolds, which allows one to list all transformation group acting transitively on flag manifolds. The fifth chapter deals with function theory on homogeneous complex manifolds and contains, amongst other things, a proof of the Matsushima-Onishchik characterization of homogeneous Stein manifolds and a description of observable subgroups of arbitrary connected linear algebraic groups which depends on tools from geometric invariant theory. There is also a section on the relationship between invariant plurisubharmonic functions and geodesic convexity.

The book ends with five sections of concluding remarks. These remarks contain some statements of results, usually without any indication of proofs. We feel they are extremely useful, as they provide the reader with some hints about related topics concerning Lie group actions in complex analysis, and each of these sections could even form a chapter in some book! A few of these have been touched on in this review. For example, using an idea of W. Kaup [10], the author notes that $\mathbb{C}^n - S$, where S is a particular discrete set with more than one element and $n > 1$, is homogeneous but does not admit any transitive action of a Lie group. The second section contains some results concerned with the existence of hypersurfaces in homogeneous complex manifolds. The problem of determining where hypersurfaces come from in the homogeneous setting is one of the most fascinating problems in the subject, in our opinion, but we forego any details here for lack of space. By the way, such questions are also related to when a homogeneous complex manifold is Kähler and to when it admits nonconstant meromorphic functions. There are also sections containing remarks about certain almost homogeneous complex spaces, symmetric spaces in the sense of Borel, and equivariant complex extensions of proper actions.

This book is well written. The style of writing is very precise, and the subject matter should be readily accessible to nonexperts. Many of the chapters can be read independently of one another, and there are ample examples presented which should allow the reader to understand what is going on. We noted a few misprints. The formula near the top of page 99 has an n missing on the RHS, and there are a couple of misquotes from the bibliography ([Gi] on page 182 should read [Gi1], and [AGi] on page 185 should read [AGin]). The eighth line on page 125 should read: *type* $G_2 : c_1 = 5,\ c_2 = 3$. As well, in the last sentence in the first section of the concluding remarks the word "real" can be deleted; see [6].

Nowadays there are a number of excellent basic books on Lie groups, Lie algebras, and Lie group actions in the differentiable setting: translations of the various books and seminar notes of A. L. Onishchik spring to mind. This book by Akhiezer is a most welcome addition on the subject of Lie group actions in complex analysis. But many important topics could not be included because of the sheer vastness of the field; e.g., the important role of the moment map and the equivariant Oka principle on Stein spaces due to Heinzner and Kutzschebauch [7] are just a couple of these. One can only hope that someday some enterprising soul will fill this gap!

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