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Nonlinear nonlocal equations in the theory of waves, by P. I. Naumkin and I. A. Shishmarev, Transl. Math. Monographs, vol. 133, Amer. Math. Soc., Providence, RI, 1994, x+289 pp., $149.00, ISBN 0-8218-4753-X

Some native peoples of the North are reluctant to mention the ocean by name, reasoning that it is not wise to presume to refer off-handedly to something that is so big. The vast topic of wave motion should perhaps be approached in the same spirit. Following the lead of the book under review, let us confine our attention here to a few human-sized features of the subject.

The simplest wave equation is

$$u_t + c_0 u_x = 0, \tag{1}$$

which has the general solution $u(x,t) = f(x - c_0 t)$, describing the steady motion of an initial profile $u(x,0) = f(x)$ to the right with speed c_0. Suppose now equation (1) is modified to read

$$u_t + c_0 u_x + u u_x = 0. \tag{2}$$

A solution of (2) will then be determined from its initial profile by the implicit equation $u = f(x - (c_0 + u)t)$, which expresses the fact that the level curves of u in the x-t plane are lines, each having slope equal to the corresponding value of $1/(c_0 + u)$. Thus if the highest point on the graph of the initial function $f(x)$ lies, say, at a height h above the x-axis, it will appear to move with the velocity $c_0 + h$. In other words, the addition of the nonlinear term uu_x to (1) makes the speed of propagation of waves dependent on their amplitudes. Moreover, since parts of the graph of f lying at different heights above the x-axis will be propagated at different velocities, the wave profile progressively distorts as time goes on (although its amplitude remains the same). An example of this type of wave motion in a natural setting occurs when unidirectional density waves are created in a gas-filled tube: the model equation is (2) with uu_x replaced by $Q(u)_x$ where $Q(u)$ is a certain function of u, and the same analysis as above will apply [W, Chapter 6].

A different effect on wave propagation, which is just as basic but a little harder to describe, occurs when equation (1) is changed to read

$$u_t + c_0 u_x + u_{xxx} = 0. \tag{3}$$

To solve the linear equation (3) one may first find a family of simple solutions and then add or superpose them to obtain more general solutions. Appropriate simple solutions are

$$Ae^{ikx}e^{i(k^3 - c_0 k)t},$$

which vary sinusoidally in x with wave length determined by the real number k and amplitude determined by the real or complex number A. Superposition then gives the general solution

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{ikx}e^{i(k^3 - c_0 k)t}\, dk, \tag{4}$$

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in which $A(k)$ is now a function of k. At time $t = 0$, (4) gives

$$u(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\, dk,$$

which shows that $A(k)$ can be interpreted as giving the amplitude distribution, over the space of parameters k, of the sinusoidal or *Fourier* components of the initial profile. Rewriting (4) in the form

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{ik[x-(c_0-k^2)t]}\, dk,$$

one sees that as t increases, the Fourier components of u remain undiminished in amplitude but propagate with velocities $c_0 - k^2$ that are dependent on their wavelengths. This behavior, known as *dispersion*, is a familiar natural phenomenon: it is responsible, for example, for the separation of the different wavelengths of a light beam as it passes through a glass prism.

The effects of dispersion on solutions of (3) are not as easy to visualize as the nonlinear effects modelled by (2), but careful analysis of the behavior of the oscillatory integral (4) will determine them clearly in the asymptotic limit as $t \to \infty$. In general, $u(x,t)$ will come to resemble the *self-similar* solution of (3) given by

$$u_S(x,t) = Bt^{-1/3}g((x-c_0t)/t^{1/3}), \tag{5}$$

in the sense that as $t \to \infty$, $|u - u_S|$ will tend to zero uniformly in x at a rate faster than $t^{-1/3}$. In (5), B is determined by the initial data through $B = \int_{-\infty}^{\infty} u(x,0)\, dx$, while g is a universal function, independent of u (for details see [D3]). Thus, dispersion acts here to reduce the initial profile to a universal shape which, as it travels with velocity c_0, is slowly shrinking in amplitude while spreading out and flattening.

Equations (2) and (3) are representative of two types of wave equations, hyperbolic and dispersive, which together describe most known wave motions [W]. Despite its simplicity, equation (2) already exhibits the most important feature of nonlinear hyperbolic equations: namely, the formation of *shock waves*. If (2) is solved with an initial profile $u(x,0)$ which is positive and decreasing in some region, then this part of the profile will gradually steepen as it propagates until it finally develops a point where the slope is vertical and the wave is said to have broken. Beyond the time of breaking, no solution of (2) exists, if the derivatives appearing in (2) are to be interpreted in the classical sense. Therefore in order to model physical phenomena it is necessary to generalize the notion of solution to include nondifferentiable and even discontinuous functions, which describe the propagation of sharp jumps in physical quantities. The question of how to define such generalized solutions, and whether they exist on extended time intervals, is a central concern of the theory of nonlinear hyperbolic waves; in general the question is very difficult, but steady progress has been made on various specific systems of interest (see, e.g. [K], [S]).

A typical nonlinear dispersive wave equation is the Korteweg-deVries equation

$$u_t + c_0u_x + uu_x + u_{xxx} = 0, \tag{6}$$

derived in 1895 as a model for water waves. In (6) the profile-steepening effects of (2) struggle against the profile-flattening effects of (3), with the result that steady travelling-wave solutions $u(x,t) = \phi_C(x-(c_0+C)t)$ are possible. Here the increment

C in the wavespeed may be any positive number, and the function $\phi_C(\xi)$ has the form of a single positive hump, with wavelength and amplitude which depend on C. A general solution $u(x,t)$ of (6) will, as $t \to \infty$, resolve into several solitary waves with different wavespeeds. Away from these solitary waves, a variety of behavior is possible: for example in some regions of x-t space, a different sort of nonlinearity-dispersion balance will force $u(x,t)$ to resemble a self-similar solution of the form

$$w(x,t) = t^{-2/3}\eta((x - c_0 t)/t^{1/3}), \tag{7}$$

(where η depends nonlinearly on the initial profile). A complete description of the asymptotics of (6) appears in [AS] and [DVZ]; the analysis there depends on a remarkable method discovered in the 1960's for obtaining solutions of (6) in explicit form and is much more complicated and subtle than the analysis of (2) or (3).

When solutions of (6) are compared to experimental measurements of actual water waves, good qualitative agreement is obtained, but in order to get good quantitative agreement it is necessary to include *dissipative* effects, or effects related to loss of energy. A simple dissipative equation is

$$u_t + c_0 u_x - u_{xx} = 0, \tag{8}$$

whose general solution is

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{ik[x - c_0 t]} e^{-k^2 t}\, dk. \tag{9}$$

Here all Fourier components of u decay in amplitude to zero as t increases, with small-wavelength components decaying most rapidly. (Note that if the coefficient of u_{xx} in (8) had been positive, the Fourier components of u would have been amplified as t increases; this is called *anti-dissipation.*) An asymptotic analysis of the integral (9) reveals that for large t, $u(x,t)$ will resemble the self-similar solution

$$B(4\pi t)^{-1/2} e^{-(x - c_0 t)^2/4t}, \tag{10}$$

where $B = \int_{-\infty}^{\infty} u(x,0)\, dx$ as in (5).

In a given wave equation, the effects of nonlinearity, dispersion, dissipation, and antidissipation may all be present together, producing results that are often hard to predict. Will dispersive and/or dissipative effects prevent nonlinear waves from breaking, so that smooth classical solutions exist for all time? Will the nonlinear effects be strong enough to at least keep discontinuities in the derivatives present for extended time intervals? In regions where the solution decays as $t \to \infty$, will dispersive effects win out to produce the profile (5), will dissipative effects dominate to produce (10), or will nonlinearity retain some influence as in (7)?

The book under review here is a compilation of answers to the above questions and others like them, based largely on the authors' own work over a period of about a decade. In some respects it is a response to Whitham's text [W], but it is a very different kind of book from Whitham's: whereas Whitham emphasizes overarching principles, formal methods for deriving model equations, and formal analyses of these equations, the present book is a rigorous and detailed mathematical treatise which concentrates on answering specific questions definitively. The arguments presented are consistently pushed to their maximum level of generality; but even the fact that some of the arguments were made to work at all is a tribute to the authors' technical powers. Fortunately, pains have obviously been taken to present the proofs clearly, so that with a little effort the reader can see how the gears mesh together to perform their appointed tasks—while nevertheless wondering if he would ever have been able to construct such a machine himself.

The term *nonlocal* in the book's title deserves some explanation. A convenient form for a general equation incorporating the various effects described above is

$$u_t + uu_x + T[u] = 0. \tag{11}$$

In the above examples, T was a differential operator and produced dispersive or dissipative effects. More generally, one could consider operators T defined by their multiplicative action on Fourier components; i.e., if

$$g(k) = \int_{-\infty}^{\infty} A(k)e^{ikx}\ dk,$$

then

$$T[g](x) = \int_{-\infty}^{\infty} A(k)m(k)e^{ikx}\ dk,$$

where $m(k)$ is a function called the *multiplier* or *symbol* associated with T. The real part of $m(k)$ will contribute to dissipative or anti-dissipative effects, while the imaginary part will create dispersive effects. For example, the dispersive-dissipative equation

$$u_t + c_0 u_x + uu_x - u_{xx} + u_{xxx} = 0, \tag{12}$$

would correspond to (11) with $m(k) = -k^2 + i(c_0 k - k^3)$.

In general, if $m(k)$ is a polynomial, then T will be a constant-coefficient differential operator; while if $m(k)$ is not a polynomial, then T will be nonlocal, in the sense that changing the values of the function $g(x)$ at points x in an open set U will affect the values of $T[g]$ at points outside U. The authors state most of their theorems for equations at the level of generality of (11), without restricting $m(k)$ to be a polynomial. However, the technical difficulties they overcome are not due to the nonlocality of the operator $T[u]$ but rather to the fact that nonlinear terms such as uu_x and (local or non-local) dispersive/dissipative terms such as $T[u]$ cannot both be given simple representations simultaneously. For example, even though equation (12) is a local equation, analysis of the effects of the dispersive and dissipative terms is complicated by the unwieldy form of the Fourier representation of uu_x:

$$uu_x = \int_{-\infty}^{\infty} e^{ikx} \int_{-\infty}^{\infty} B(k-l)(il)B(l)\ dl\ dk,$$

where

$$u(x,t) = \int_{-\infty}^{\infty} B(k,t)e^{ikx}\ dk.$$

This particular fact requires the authors to invent an elaborate and ingenious method to keep track of the contribution of the nonlinear term in their study of the long-time asymptotics of (12).

Chapters 1 through 5 of the book are mostly concerned with the complementary questions of whether smooth solutions exist globally and whether waves with smooth initial profiles will eventually break. Included among the general results here is one which settles a conjecture in [W]. Whitham had noted that solutions of the model equation (6) did not break as physical water waves do and had conjectured that breaking solutions would exist for equation (11) with $m(k) = \sqrt{\frac{\tanh k}{k}}$, which describes more accurately than (6) the wavelength-speed relation imposed by the full equations of motion for water waves. Here the authors show that a solution of Whitham's proposed equation will indeed break if the slope of its initial profile is sufficiently large and negative at some point; moreover they are able to give a fairly narrow range of times (depending on the steepness of the initial profile) within which the breaking must occur.

Chapters 6 through 10 treat the long-time asymptotic behavior of waves in which dissipative effects eventually dominate. The results in these chapters fit into a general landscape recently delineated by Dix [D2], [D3], in which the predominance of nonlinear, dispersive, or dissipative effects can be heuristically predicted from the form of the wave equation and the behavior near $k = 0$ of the Fourier transform $A(k)$ of the initial wave profile. For generic solutions of equation (12) with integrable initial profiles, dissipation will balance with nonlinearity as $t \to \infty$ in such a way that the asymptotic form will be a self-similar solution of Burgers' equation

$$u_t + c_0 u_x + u u_x - u_{xx} = 0$$

(cf. [ABS], [D1]). The results in the present book apply to (12) in the special case when $\int_{-\infty}^{\infty} u(x, 0)\, dx = 0$ (so that $A(0) = 0$). In this case, as the authors show in Chapter 7, dissipation dominates nonlinearity (at least for small initial data), with the result that $\int_{-\infty}^{x} u$ will have the asymptotic form (10). (Note: this result is misquoted later in Chapter 9.) Since their method of proof does not depend on the specific form of equation (12), the authors can also use it with straightforward modifications in a number of other interesting situations. In fact, the method is used first in Chapter 6 to handle the semilinear heat equation

$$u_t - F(u) - u_{xx} = 0,$$

without the restriction on the integral of $u(x, 0)$ that is necessary for (12).

As a history of the fruitful collaboration of two mature mathematicians over an extended period, the book naturally contains a good deal more material than the preceding examples indicate. It is very readable, so much so that the reader can easily spot and correct the large number of misprints (which are perhaps inevitable in a book whose ratio of symbols to text hovers near unity). It will be a source of inspiration not only to those who study nonlinear wave equations, but to all who like to see hard problems solved by masterful use of basic techniques.

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