

Introduction to symplectic topology, by D. McDuff and D. Salamon, Oxford Science, 1995, viii + 425 pp., \$82.00, ISBN 0-19-851177-9

A C^∞ manifold M may be endowed with several additional structures. One of the most natural is a Riemannian metric, leading to Riemannian geometry. Another is a complex structure, so that a manifold M of real dimension $2m$ becomes a complex manifold which may be studied using complex geometry: the charts in an atlas are identified with open subsets of \mathbb{C}^m , and the composition of chart maps is assumed to be a holomorphic map between domains in \mathbb{C}^m . A third very natural structure is a *symplectic* structure, which is a nondegenerate closed 2-form ω on M . These three types of geometrical structures intersect in *Kähler geometry*: any two of the three geometrical structures which are compatible in the natural way determine the third, and the three compatible structures constitute a Kähler structure on the manifold.

The roots of symplectic geometry are as old as the other two branches of geometry mentioned above, since it originated as the natural mathematical framework for classical mechanics; however, until recently symplectic geometry had a somewhat lower profile in the mathematical landscape than Riemannian or complex geometry. This situation is rapidly changing, and symplectic geometry is a swiftly expanding field attracting increasing attention from researchers and students alike; for some time there has been an urgent need for a comprehensive and authoritative textbook, such as has existed for decades in Riemannian geometry, that could be used as a point of departure for graduate courses. McDuff and Salamon's book admirably fills this need: two years beyond its publication it has already become one of the definitive references in the field. It belongs on the bookshelf of students and researchers whose work involves any branch of geometry and in the library of any research university.

Although a symplectic form is a geometrical structure, many problems that arise when one posits the existence of a symplectic structure reduce to problems in topology; hence the book's title is appropriate, since on the whole its emphasis is on the topological aspects of the subject. Nonetheless the book provides a broad and comprehensive introduction to all aspects of the subject, including the basics: it assumes no prior familiarity with symplectic structures. The book provides ample cross-referencing: though it is quite self-contained, it nonetheless provides extensive references to the surrounding literature. One topic that is not treated is pseudoholomorphic curves in symplectic manifolds; for this material the authors refer readers to their own monograph [13] or to the earlier collection of articles [3].

The origins of symplectic geometry are in classical mechanics. The most natural example of a symplectic manifold is \mathbb{R}^{2m} , the *phase space* (in other words the space parametrizing the position and momentum of a system with m degrees of freedom):

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the symplectic form is simply

$$(1) \quad \omega_0 = \sum_{i=1}^m dp_i \wedge dq_i,$$

where the position variables of the system are q_i and the corresponding momenta are p_i . One of the fundamental results of the subject is the Darboux theorem, which tells us that locally one can find coordinates in which any symplectic form is given in this way. Weinstein's proof [19] of the Darboux theorem is presented in Chapter 3: this proof makes use of an argument due to Moser (often referred to as "Moser's method"), which allows one (given a family $\{\omega_t, t \in [0, 1]\}$ of 2-forms satisfying appropriate hypotheses which interpolates between a given symplectic form ω_1 and the standard Darboux symplectic form ω_0 given by (1)) to construct a family of diffeomorphisms ϕ_t (for $t \in [0, 1]$) for which $\phi_t^* \omega_t = \omega_0$, and which thereby exhibits a diffeomorphism under which the form ω_1 pulls back to the standard form ω_0 .

For any C^∞ function $H : M \rightarrow \mathbb{R}$ on a symplectic manifold (M, ω) , the equation

$$dH(\cdot) = \omega(X_H, \cdot)$$

defines a vector field X_H (the *Hamiltonian vector field*) associated to H . A vector field is thus associated to any smooth function on a symplectic manifold. The flow associated to this vector field,

$$(2) \quad \frac{dx}{dt} = X_H(x(t)),$$

is called the *Hamiltonian flow*. One may generalize this by considering the flow

$$(3) \quad \frac{dx}{dt} = X_{H_t}(x(t))$$

given by the family of vector fields X_{H_t} associated to a family of functions H_t for $t \in [0, 1]$ (*time-dependent Hamiltonian flow*). The prototypical example of Hamiltonian flow comes from the usual Hamiltonian H (the sum of the kinetic energy and the potential energy) on the phase space \mathbb{R}^{2m} : for instance, for a harmonic oscillator potential (governing the motion of a mass on a spring) we have

$$H = \frac{1}{2} \sum_i (p_i^2 + q_i^2).$$

The equation for Hamiltonian flow then encodes the equations of motion of the system (*Hamilton's equations*). Readers interested in pursuing the connection of symplectic geometry with mechanics will find the book of Arnol'd [1] valuable.

One important area in symplectic geometry treats group actions which preserve the symplectic structure and the appropriate definition of a quotient of a symplectic manifold by such a group action which is still a symplectic manifold: one must introduce *moment maps* (Hamiltonian functions whose Hamiltonian vector fields are the vector fields generated by the group action), and then the *symplectic quotient* or *Marsden-Weinstein reduction* is defined as the quotient of a level set of the moment map by the group action. Under appropriate hypotheses the symplectic quotient inherits a symplectic form from that on the original symplectic manifold M . This material is treated in Chapter 5; readers interested in pursuing these topics in more detail will wish to consult the books of Audin [2], Berline, Getzler and Vergne [4], Guillemin and Sternberg [7], and Guillemin, Lerman and Sternberg [6].

A variety of other subjects are treated in the course of the book. For example, in Chapter 6 there is an excellent presentation of symplectic blowups (which are closely related to blowups in algebraic geometry, though different in some important respects). This is followed by a sketch of Gompf's important construction of families of symplectic four-manifolds whose fundamental groups include all finitely presented groups. Chapter 10 treats the structure of the group of diffeomorphisms preserving the symplectic structure of a manifold: this group is infinite-dimensional, unlike the group of isometries of a compact Riemannian manifold which is always a compact Lie group.¹ Several important, very recent advances in symplectic geometry are mentioned though not treated in any detail. One is Donaldson's construction of a symplectic submanifold representing the Poincaré dual of a multiple of the symplectic form, which is discussed briefly in Chapter 4. Another is Taubes's recent work on symplectic 4-manifolds using Seiberg-Witten invariants, which establishes in particular that the symplectic structure on complex projective space $\mathbb{C}P^2$ is unique, and also enlarges the class of four-manifolds which are known to have no symplectic structure.

The heart of the book is Part IV, which consists of two chapters, each devoted to one of the central topics in symplectic topology over the last few years. The first chapter of Part IV treats the Arnol'd conjecture, while the second treats symplectic capacities. The Arnol'd conjecture (which had at the time of publication of the present volume been established for a large class of symplectic manifolds)² concerns the problem of determining a lower bound on the number of fixed points of a symplectic diffeomorphism arising from a time-dependent Hamiltonian flow. Arnol'd conjectured that such a diffeomorphism must have at least as many fixed points as the minimal number of critical points of a Morse function on M . This conjecture was proved by Floer for a fairly large class of symplectic manifolds (*monotone symplectic manifolds*); Floer's proof used the gradient flow of the *symplectic action functional*. The symplectic action functional is a function on the (infinite-dimensional) space of C^∞ maps from S^1 to M which takes values in \mathbb{R}/\mathbb{Z} : its value at a loop γ in M may be defined as the symplectic area of a disk whose boundary is γ . (The symplectic action functional is the natural analogue of the *Chern-Simons functional* on connections on a bundle on a three-manifold, which Floer used to define *instanton homology* or *Floer homology*.) McDuff and Salamon present a proof of the Arnol'd conjecture for the standard torus, in which they replace the symplectic action functional by a discrete analogue which is defined on a finite-dimensional space. An earlier survey on this material is [14].

The final chapter of Part IV treats *symplectic capacities*, which are an essential tool in symplectic topology. (Readers wishing a less detailed introduction may find it helpful to look at the beginning sections of the recent survey by Lalonde [10], and at Viterbo's Séminaire Bourbaki notes [16].) Symplectic capacities may be used to address the question of which symplectic manifolds may be symplectically embedded in other symplectic manifolds of the same dimension.

Obviously the symplectic volume $\int_M \omega^m/m!$ provides an obstruction to symplectic embedding: in dimension 2 this is the only obstruction, but in higher dimensions

¹This was established more than fifty years ago by Myers and Steenrod.

²Very recently proofs of the Arnol'd conjecture in full generality have been given by several groups, including Fukaya and Ono, Liu and Tian, and Ruan [5, 12, 15]. An additional proof has been announced by Hofer and Salamon [8].

more subtle obstructions exist. Gromov defined the *Gromov width* w_G of a symplectic manifold M , which is the area of a disk whose radius equals that of the largest ball that can be symplectically embedded in M . He proved the *nonsqueezing theorem*, which tells us that one cannot symplectically embed the symplectic ball $B^{2m}(r)$ of dimension $2m$ and radius r into the symplectic cylinder $B^2(R) \times \mathbb{R}^{2m-2}$ unless $r \leq R$. Gromov's proof of the nonsqueezing theorem used pseudoholomorphic curves, but there are several alternative proofs.³ McDuff and Salamon follow Hofer and Zehnder in using variational methods to construct a *symplectic capacity*, which is a functional c which assigns a nonnegative number to every symplectic manifold in such a way that certain axioms are satisfied:

- a. *Monotonicity*: If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds of the same dimension and if there is a symplectic embedding of M_1 into M_2 , then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.
- b. *Conformality*: If λ is a positive real number, then $c(M, \lambda\omega) = \lambda c(M, \omega)$.
- c. *(Weak) Nontriviality*: The capacity of the symplectic ball $B^{2n}(R)$ is > 0 while that of the symplectic cylinder $B^2(R) \times \mathbb{R}^{2n-2}$ is $< \infty$.

McDuff and Salamon construct the capacity c_{HZ} (the *Hofer-Zehnder capacity*) that was defined by Hofer and Zehnder using properties of periodic orbits: it has the property that

$$c_{HZ}(B^{2m}(r)) = c_{HZ}(B^2(r) \times \mathbb{R}^{2m-2}),$$

from which the nonsqueezing theorem follows immediately. As another application of the Hofer-Zehnder capacity, McDuff and Salamon use it to give a proof of a celebrated conjecture of Weinstein concerning periodic orbits of vector fields on appropriate classes of hypersurfaces in symplectic manifolds.

Several different types of symplectic capacities have been defined which are not equivalent to each other: for instance the Gromov width w_G described above satisfies the axioms of a symplectic capacity and is always less than or equal to the Hofer-Zehnder capacity c_{HZ} . Hofer and Zehnder originally constructed the capacity c_{HZ} by applying variational techniques to the symplectic action functional. As in their treatment of the Arnol'd conjecture, McDuff and Salamon have chosen to avoid the use of the infinite-dimensional analytical techniques used in the original proof by constructing a finite-dimensional analogue of the symplectic action functional which can be treated by finite-dimensional variational methods. This makes the material accessible to readers uncomfortable with analytical techniques using infinite-dimensional Banach spaces; those readers fluent in the language of infinite-dimensional analysis will find Hofer and Zehnder's original argument presented in Chapter 3 of [9].

This book has already earned its place as a basic reference for workers in the field; it will also make the task of beginning research substantially simpler for graduate students starting work in this area. The presence of a text of this type opens a field up to graduate students, since it makes the prerequisites to research much more accessible than the original references did. Researchers who learned their trade before a reference of this type was available have good cause to envy those

³The proof using pseudoholomorphic curves was extended by Lalonde and McDuff to give results for more general symplectic manifolds: see the survey [10] and the original article [11]. A third proof differing both from Gromov's original approach and from the proof presented here by McDuff and Salamon is the proof given by Viterbo [17] using *generating functions*; background on these is provided in Chapter 9 of the present volume.

beginning work now that an authoritative and comprehensive reference of this type is available. McDuff and Salamon have done an enormous service to the symplectic community: their book greatly enhances the accessibility of the subject to students and researchers alike.

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