# BOOK REVIEWS

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*$C^*$-algebras by example*, by Kenneth R. Davidson, Field Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996, xiv+309 pp., \$59.00, ISBN 0-8218-0599-1

A locally compact Hausdorff space $X$ is completely determined by its associated Banach algebra $C(X)$, consisting of all complex-valued continuous functions vanishing at infinity, endowed with the sup norm. We recover the set $X$ as the set of homomorphisms from $C(X)$ into $\mathbb{C}$ (the point evaluations), and the topology of $X$ reappears on the set of homomorphisms by restricting the weak*-topology of the dual space of $C(X)$. In its most abstract form, the spectral theorem asserts that the algebras $C(X)$ are characterized as the commutative $C^*$-algebras. This is the first of several reasons why operator algebraists like to think of noncommutative $C^*$-algebras as if they were 'noncommutative spaces', even when there is no ordinary space at all.

$C^*$-algebras (including noncommutative ones) are distinguished from other complex Banach algebras by their symmetry: there is an involution $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and which is related to the norm by the elegant condition

$$\|x^*x\| = \|x\|^2.$$

The most concrete examples of $C^*$-algebras occur as complex linear subalgebras of the algebra $\mathcal{B}(H)$ of all bounded linear operators on a complex Hilbert space $H$, which are closed in the norm topology of $\mathcal{B}(H)$ and which are stable under the natural involution $T \mapsto T^*$ of $\mathcal{B}(H)$. Nowadays, most mathematicians have heard of the Gelfand-Naimark theorem, which asserts that every $C^*$-algebra is isometrically $*$-isomorphic to a concrete $C^*$-algebra of operators.

Somewhere in school we learn that peculiar equations like $x^2+1=0$ can actually be solved. Solving them amounts to constructing a world where things like $\sqrt{-1}$ make sense, and showing that this world contains all solutions of the given equation. $C^*$-algebras provide powerful tools and a natural context for dealing with noncommutative phenomena, including the solution of peculiar equations. To illustrate the point, consider Heisenberg's fundamental equation which underlies the uncertainty principle of quantum mechanics

$$pq - qp = i\mathbf{1}. \tag{1}$$

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One may imagine that $\mathbf{1}$ is the unit in some complex algebra $A$ with involution, that $p, q$ are elements of $A$ satisfying $p^* = p$, $q^* = q$, and $i = \sqrt{-1}$. We want to find all solutions of (1), and here the mystery is that $pq$ is not the same as $qp$.

Experience with elementary linear algebra might lead one to seek solutions of (1) in the $*$-algebra of all $n \times n$ complex matrices. However, such solutions do not exist because the trace of the matrix on the left of (1) would be zero while the trace of the matrix on the right would be nonzero. In fact, it is impossible to solve (1) using bounded operators $p$, $q$ on any infinite dimensional space, or more generally in any Banach algebra.

Retreating only slightly, we can regard (1) as an infinitesimal equation holding in some Lie algebra, and we look for its group. If one formally exponentiates $p$ and $q$ to obtain two one-parameter groups

$$U_t = e^{itq}$$
$$V_t = e^{itp},$$

then the equations $q^* = q$ and $p^* = p$ translate into $U_t^* = U_{-t}$ and $V_t^* = V_{-t}$, $t \in \mathbb{R}$. More significantly, after a bit of computation one finds that (1) turns into Weyl's commutation relation

$$V_t U_s = e^{-ist} U_s V_t \qquad s, t \in \mathbb{R}. \tag{2}$$

One can use the relations (2) to generate a Banach algebra consisting of "continuous" formal linear combinations. Thus with every integrable function $f : \mathbb{R}^2 \to \mathbb{C}$ we associate the formal integral

$$W_f = \int_{\mathbb{R}^2} f(s,t) U_s V_t \, ds \, dt.$$

Given two integrable functions $f$ and $g$, their product $f \cdot g$ is defined as the unique integrable function satisfying $W_f W_g = W_{f \cdot g}$. Using (2), one easily carries out the formal computation to discover how $f \cdot g$ should be defined, and after another patient calculation one finds that $f \cdot g$ is a phase-shifted convolution

$$f \cdot g(x,y) = \int_{\mathbb{R}^2} f(s,t) g(x-s, y-t) e^{-it(x-s)} ds \, dt.$$

A similar computation identifies an involution in this Banach algebra $L^1(\mathbb{R}^2, \cdot)$ after noting that

$$(U_s V_t)^* = V_t^* U_s^* = V_{-t} U_{-s} = e^{-ist} U_{-s} V_{-t}.$$

Thus $L^1(\mathbb{R}^2, \cdot)$ becomes a Banach algebra with an isometric involution. This Banach algebra is not a $C^*$-algebra because we do not have $\|f^* \cdot f\| = \|f\|^2$. However, in such situations there is always a largest $C^*$-algebra seminorm $\| \cdot \|_0$ which satisfies $\|f\|_0 \le \|f\|$ for every $f$. The completion of $L^1(\mathbb{R}^2, \cdot)$ in the seminorm $\| \cdot \|_0$ is a $C^*$-algebra $\mathcal{A}$ which has the same representation theory as $L^1(\mathbb{R}^2, \cdot)$.

The remarkable fact is that $\mathcal{A}$ is isomorphic to the $C^*$-algebra $\mathcal{K}$ of all compact operators acting on a separable Hilbert space. That follows from the fact that there is an integrable function $e$ having the two properties

$$e\mathcal{A}e = \{\lambda e : \lambda \in \mathbb{C}\}$$
$$\mathcal{A} = \overline{\operatorname{span}}\{xey : x, y \in \mathcal{A}\}$$

(in fact, $e$ has the form $e(s,t) = e^{p(s,t)}$ for an appropriate quadratic polynomial $p$). These two properties alone serve to identify the structure of $\mathcal{A}$; the first implies that $e$ is a scalar multiple of a rank-one projection and the second implies that $\mathcal{A}$ is spanned by its rank-one operators.

$\mathcal{K}$ is the most elementary example of an infinite dimensional *simple* $C^*$-algebra, that is, one which has no nontrivial norm-closed two-sided ideals ($\mathcal{K}$ has many ideals, but the nontrivial ones are not closed in the operator norm). It has the fundamental property that any two of its irreducible representations are unitarily equivalent, and more generally, every representation is unitarily equivalent to a direct sum of copies of any irreducible one. It follows that we have a complete description of all pairs $(U, V)$ of one-parameter unitary groups which satisfy (2). On the Hilbert space $L^2(\mathbb{R})$, for example, one can define

$$U_s f(x) = e^{isx} f(x), \qquad V_t f(x) = f(x - t). \tag{3}$$

One sees immediately that (2) is satisfied, and this particular pair $(U, V)$ is called the Schrödinger representation of the canonical commutation relations. It is not hard to show that this representation is irreducible. The corresponding solution of (1) is given by the pair $(q, p)$ of self-adjoint generators of the groups $U$ and $V$

$$qf(x) = xf(x), \qquad pf = -i\frac{df}{dx}, \tag{4}$$

defined on their appropriate domains. Moreover, the preceding remarks imply that any pair $(U, V)$ of one-parameter unitary groups which satisfy (2) can be realized (after a change in coordinates) by replacing scalar valued functions in $L^2(\mathbb{R})$ with vector valued functions $f : \mathbb{R} \to E$, $E$ being some Hilbert space whose dimension is equal to the multiplicity of the corresponding representation of $\mathcal{A}$. The definition of $U$ and $V$ in this space of vector functions is the same. This is the one-dimensional case of the Stone-von Neumann theorem, the cornerstone of the mathematical development of quantum mechanics [12].

Equation (1) is associated with one-dimensional quantum systems. The situation for $n$-dimensional quantum systems is similar, and again one arrives at an enveloping $C^*$-algebra that is isomorphic to $\mathcal{K}$. More generally, any system of noncommutative equations that makes sense in an algebra with involution, and which has the property that any two "irreducible representations" of the equations are unitarily equivalent, will be associated with a $C^*$-algebra that turns out to be isomorphic to $\mathcal{K}$ (I have glossed over minor details concerning cardinality and the control of norms in order to make the essential point).

On the other hand, when there are infinitely many degrees of freedom, both the commutation relations and the anticommutation relations lead to simple $C^*$-algebras which are not isomorphic to the compact operators. Such algebras have many (uncountably many) irreducible representations which are mutually inequivalent, and worse, the space of equivalence classes of irreducible representations cannot be parameterized in a measurable way. In such cases one must give up the idea of finding *all* solutions of the given equations. Instead, one attempts to understand the equations in terms of the structure of their associated $C^*$-algebras, on $C^*$-algebraic terms. For example, the Cuntz algebra $\mathcal{O}_2$ is the $C^*$-algebra generated by a pair of elements $u, v$ satisfying

$$u^*u = v^*v = \mathbf{1}, \quad uu^* + vv^* = \mathbf{1}.$$

$\mathcal{O}_2$ is a famous example of a simple $C^*$-algebra whose representation theory is "bad" in the sense described above. On the other hand, it has a particularly revealing irreducible representation, its structure is elegant and amenable to analysis, we know its noncommutative algebraic topology (that is, its $K$-theory and $Ext$-theory), and it is just now coming to be recognized as central to understanding the structure of a broad class of simple $C^*$-algebras.

Many problems in mathematics lead almost inevitably (with the benefit of contemporary vision) to noncommutative $C^*$-algebras. Whenever one has a dynamical system or a foliation of a manifold, there is an associated $C^*$-algebra which is always noncommutative and often simple. The Kronecker foliation of the torus appears somewhat intractable from the classical (commutative) point of view, but when one looks carefully at its $C^*$-algebra, one finds a rich and beautiful structure [2]. Following its development by Cuntz, Kasparov's $KK$-bifunctor has provided a *concrete* unification of $K$-homology with $K$-cohomology that does not exist in the world of spaces and maps into matrices.

Several books have appeared recently that relate specifically to $C^*$-algebras. Along with the one under review there are books of Fillmore ([5], a broad survey including some detail), Jensen and Thomsen ([7], a comprehensive starter volume for $C^*$-modules and $KK$-theory), Murphy ([13], basic operator algebras through $K$-theory), Wegge-Olsen ([14], $K$-theory of operator algebras done gently), in addition to Connes' masterpiece [2]. Kadison and Ringrose have completed their series on operator algebras [8], [9], [10], [11], and a new edition of the second volume of Bratteli and Robinson has appeared ([1], emphasizing the algebraic approach to quantum statistical mechanics). These take up their posts on operator algebraists' bookshelves alongside Dixmier's classics [3], [4].

The idea behind Davidson's book is a good one: one should learn about $C^*$-algebras by studying interesting examples of $C^*$-algebras...in detail. These include $AF$ algebras (including their $K$-theory), the Toeplitz and Cuntz algebras, irrational rotation algebras, group $C^*$-algebras (especially those of discrete groups), crossed products (by $\mathbb{Z}$-actions), and extensions of the compact operators by $C(X)$. All the proofs are there. This means that one can assign parts of Davidson's book to good students learning the subject and expect good results (I have done that, and have had same). The details persist even when the going gets tough, as it sometimes does. For example, in the last chapter the Brown-Douglas-Fillmore theory is completely worked out for subsets of the complex plane. While there are some simplifications of the original BDF work here, I would have preferred a more conceptual treatment, perhaps based on Higson's approach to the homotopy invariance of $Ext$ ([6]; Higson and Roe are currently writing a book on $K$-homology that will contain such a treatment, to be published by Oxford University Press). Nonetheless, this is the only book I know in which one can go through the BDF classification of essentially normal operators and follow, point by point, to the end.

I look forward to using Davidson's book the next time I teach operator algebras, perhaps complemented with Fillmore's "User's Guide" [5].

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William Arveson
University of California, Berkeley
*E-mail address*: arveson@math.berkeley.edu