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Analysis on symmetric cones, by Jacques Faraut and Adam Koranyi, Oxford University Press, London and New York, 1994, xii + 382 pp., $85.00, ISBN 0-198-53477-9

Introduction. To quote from the preface, "The purpose of this book is to give a self-contained exposition of the geometry of symmetric cones, and to develop analysis on these cones and on the complex tube domains associated with them." Who better to write such a book than two of the foremost researchers on symmetric cones and related areas of analysis and geometry? In my view, they have written a book that is significant for a number of reasons.

◦ The topics treated in the book are important. For example, the subject of analysis on symmetric cones has substantial interfaces with many areas of mathematics, including number theory, Lie theory and geometry, classical analysis, partial differential equations, operator theory, group representations, and harmonic analysis. In addition, analysis on particular families of symmetric cones such as matrix cones or Lorentz cones (see below) arises in multivariate statistics, mathematical physics, statistical mechanics, and even theoretical aspects of molecular chemistry.

◦ The book takes the reader right up to the cutting edge of current research.

◦ The book fills a void in the mathematics literature. For, in the past, one would have found it demanding and time consuming to learn this area of mathematics from scattered journal articles, lecture notes, or highly sophisticated algebraic treatises.[1] In particular, this book is the first to provide a complete and self-contained exposition of the structure theory of symmetric cones from the perspective of real and complex analysis, as well as the first comprehensive exposition of noncommutative analysis on symmetric cones and related domains that applies to *all* symmetric cones in complete generality, without recourse to the classification of symmetric cones.[2]

The reader of this review who is already conversant with the algebra and geometry of symmetric cones probably does not need a review; the preface and table of contents, together with the paragraphs that introduce each chapter and the notes at the end of each chapter, will convey the contents and flavor of the book. Thus, this review is written primarily for the practicing mathematician or statistician who knows little about symmetric cones, but who may find their study interesting and of value in research. To this end, the first several sections of this review will serve as a brief introduction to the study of symmetric cones. The goal is to pique the reader's interest in this fascinating subject.

1991 *Mathematics Subject Classification.* Primary 22E30, 32M15, 43A85, 17C10; Secondary 43A90, 32A37, 32H99, 33C50, 62H99.

[1] The geometry of symmetric cones and applications to Lie theory and analysis owes much to M. Koecher. His lecture notes [6] have been a standard reference in the subject for over 30 years. Another treatment, considerably more algebraic, is contained in the book [9] by I. Satake and is part of a comprehensive Lie theoretic treatment of bounded symmetric domains. Both are excellent references.

[2] A substantial amount, but far from all, of the analysis described in this book was first worked out for matrix cones and appears in the mathematics and statistics literature (see, for example [8] and [3]). This book extends the development to the general context.

Note: In what follows we will refer to the book under review as simply "the book". Chapter references are to the book.

Symmetric cones and Euclidean Jordan algebras [Ch. I-IV]. One should observe at the outset that the study of symmetric cones is closely related to – in fact, inseparable from – the study of a certain kind of nonassociative algebra over the real field known as a *Euclidean Jordan algebra*. Indeed, the book might well have been titled *Analysis on Euclidean Jordan algebras and related domains.* To be more specific, there is a one-to-one correspondence between symmetric cones and Euclidean Jordan algebras that will be made explicit later in the review.

A motivating example: Real matrix space. For the reader who is unfamiliar with the notion of a symmetric cone or Euclidean Jordan algebra,[3] it is useful and entirely appropriate to think of a Euclidean Jordan algebra as the conceptual generalization of the *matrix space* consisting of all real $m \times m$ symmetric matrices, in which case the associated symmetric cone is the *matrix cone* of all symmetric $m \times m$ matrices that are positive-definite.

Denote by $V = V_m = \mathrm{Sym}(m, \mathbb{R})$ the vector space of all real symmetric $m \times m$ matrices equipped with the inner product $\langle x|y\rangle = \mathrm{tr}(xy)$. Since the product of symmetric matrices need not be symmetric, V is not closed under matrix multiplication. However, the anti-commutator

$$x \circ y = \frac{1}{2}(xy + yx) \tag{1}$$

is a well-defined operation on V called the *Jordan product* or *Jordan multiplication*, with respect to which V has the structure of a commutative but (for $m > 1$) not associative real algebra. As a substitute for associativity one has the so-called *Jordan identity*

$$x^2 \circ (y \circ x) = (x^2 \circ y) \circ x. \tag{2}$$

In terms of the linear transformation $L(x)$ on V defined as Jordan multiplication by x (i.e., $L(x)y = x \circ y$), identity (2) is recast as the commutation relation

$$L(x)L(x^2) = L(x^2)L(x). \tag{3}$$

By standard properties of the trace, the linear transformation $L(x)$ on V is seen to be self-adjoint; i.e.,

$$\langle x \circ y \,|\, z\rangle = \langle y \,|\, x \circ z\rangle. \tag{4}$$

Next, denote by $\Omega = \Omega_m = \mathrm{Pos}(m, \mathbb{R})$ the subset of V of matrices that are positive-definite. Then Ω is an open convex cone in V that is *self-dual* in the sense that $\Omega = \{x \in V : \langle x \,|\, y\rangle > 0 \text{ for all } y \neq 0 \text{ in the closure of } \Omega\}$. Note that $\Omega = \mathrm{Pos}(m, \mathbb{R})$ can also be characterized as the connected component of the $m \times m$ identity matrix ε in the set of invertible elements of V.

Finally, one brings in the group theory. Let $G = \mathrm{GL}_+(m, \mathbb{R})$ be the (connected) real general linear group consisting of all real $m \times m$ matrices with positive determinant. Then each $g \in G$ acts quadratically on V by $g \cdot x = gxg^{\mathrm{t}}$, and this action is transitive on Ω . Thus, the $m \times m$ rotation group $K = \mathrm{SO}(m) = \{g \in G : gg^{\mathrm{t}} = \varepsilon \text{ and } \det g = 1\}$ is the isotropy subgroup of ε (i.e., an element $g \in G$ lies in K if

[3]What is now called a *symmetric cone* in older terminology was referred to as a *domain of positivity*, and what is now known as a Euclidean Jordan algebra was called a *formally real* Jordan algebra.

and only if $g \cdot \varepsilon = \varepsilon$), and the mapping $gK \mapsto x = g \cdot \varepsilon$ identifies the homogeneous (symmetric) space G/K with the cone Ω . In symbols,

$$\Omega \cong G/K = \mathrm{GL}_+(m, \mathbb{R})/\mathrm{SO}(m) \tag{5}$$

and we say that Ω is *homogeneous* under the action of G.

The general theory. Armed with the above example, we make the following definitions.

(i) A *Jordan algebra* over a field $\mathbb{F}$ is a commutative (but in general not associative) algebra V with identity ε that satisfies property (2). When $\mathbb{F}$ is the real field and the Jordan algebra is equipped with an inner product relative to which property (4) holds, then one says the Jordan algebra is *Euclidean.*

(ii) Let $(V, \langle \cdot \mid \cdot \rangle)$ be a finite-dimensional inner product space. A *symmetric cone* in V is a nonempty open convex cone in V that is self-dual and homogeneous under the automorphism group $G = G(\Omega)$, consisting of all invertible linear transformations of V that preserve Ω .

(iii) A Jordan algebra is *simple* if it has no nontrivial ideals, and a symmetric cone is *irreducible* if it is not the direct product of two or more symmetric cones.

Thus, $V_m = \mathrm{Sym}(m, \mathbb{R})$ and $\Omega = \Omega_m = \mathrm{Pos}(m, \mathbb{R})$ are prototypical examples of a simple Euclidean Jordan algebra and an irreducible symmetric cone, respectively. Indeed, as the following structural results attest, in general a symmetric cone is related to a Euclidean Jordan algebra in exactly the same way that $\mathrm{Pos}(m, \mathbb{R})$ is related to $\mathrm{Sym}(m, \mathbb{R})$.

The correspondence between Ω and V [Ch. III, Sections 2 and 3]. *(i)* If Ω is a symmetric cone, then the ambient space V carries the structure of a Euclidean Jordan algebra, and Ω is the connected component of the identity ε in the set of invertible elements of V. Conversely, if V is a Euclidean Jordan algebra, then the connected component of the identity in the set of invertible elements of V is a symmetric cone.

(ii) A symmetric cone is irreducible if and only if the associated Euclidean Jordan algebra is simple.

(iii) A Euclidean Jordan algebra decomposes as a direct sum of simple ideals, and a symmetric cone decomposes as a direct sum of irreducible symmetric cones.

In short, properties *(i)*-*(iii)* reduce the study of symmetric cones to the study of simple Euclidean Jordan algebras.

Classification [Ch. V]. The simple Euclidean Jordan algebras were classified by Jordan, von Neumann, and Wigner [5].[4] We state the classification in terms of the corresponding symmetric cones: Up to isomorphism, the irreducible symmetric cones fall into four families of so-called *classical* cones together with one other cone that is said to be *exceptional*. The first three families of classical symmetric cones are the *matrix cones* $\mathrm{Pos}(m, \mathbb{R})$, defined above, the cones $\mathrm{Pos}(m, \mathbb{C})$ of all complex $m \times m$ Hermitian matrices that are positive-definite, and the cones $\mathrm{Pos}(m, \mathbb{H})$ of all quaternionic $m \times m$ Hermitian matrices that are positive-definite. The fourth family of classical cones consists of the *Lorentz cones* $\Lambda_n = \{x \in \mathbb{R}^n : x_1^2 - x_2^2 - \cdots - x_n^2 > 0 \text{ and } x_1 > 0\}$, which are also referred to as the *forward light cones*. The exceptional

[4]The concept of a Jordan algebra has its roots in work of Jordan, von Neumann, and Wigner in the 1930's on the formalism of quantum mechanics in which observable quantities are represented by Hermitian operators (see [7]).

cone is a 27-dimensional cone of 3×3 "positive-definite" matrices over the Cayley algebra.

The importance of the concept of a symmetric cone, therefore, lies in the capability of studying all of the classical cones simultaneously without resorting to the above classification, and at the same time including the exceptional cone, the study of which can be extremely complicated in the context of its definition as matrices over the Cayley algebra.

Linear algebra. Familiar concepts of linear algebra that apply to the matrix space $\mathrm{Sym}(m, \mathbb{R})$ and matrix cone $\mathrm{Pos}(m, \mathbb{R})$ extend to the general context. Here are a few examples.

∘ The concepts of *minimal polynomial, trace*, and *determinant* of a matrix generalize to the context of Jordan algebras [Ch. II].

∘ The *spectral theorem* for a symmetric matrix generalizes to elements of a Euclidean Jordan algebra, and the concept of a resolution of the identity matrix (i.e., a set of orthogonal projections $e_1 \ldots e_r$ that sum to the identity matrix) generalizes to what is called a *Jordan frame*. The number of elements r in a Jordan frame is the *rank* of the Jordan algebra or cone [Ch. III]. (Hence, the rank of $\mathrm{Sym}(m, \mathbb{R})$ is m.)

∘ The *polar decomposition* in a simple Jordan algebra V is a generalization of the diagonalization of a symmetric matrix by an orthogonal matrix. Namely, if one fixes a Jordan frame $e_1 \ldots e_r$, then any element $x \in V$ decomposes as $x = k \cdot (\lambda_1 e_1 + \cdots + \lambda_r e_r)$ where $k \in K$ and K is the isotropy subgroup of ε in $G = G(\Omega)$, and the real numbers $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of x. The *rank* of x is the number of nonzero eigenvalues, and x is *regular* if the rank is r [Ch. VI].

∘ The *Peirce decomposition* with respect to a Jordan frame, when specialized to the case of $\mathrm{Sym}(m, \mathbb{R})$, is the usual basis of matrix units E_{ij} for $1 \leq i \leq j \leq m$ [Ch. IV].

∘ The *Gauss decomposition* for an element of a symmetric cone reduces in the case of a positive-definite symmetric matrix x to the decomposition $x = \ell\ell^{\mathrm{t}}$ with ℓ lower triangular [Ch. VI].

∘ In establishing the Gauss decomposition for Euclidean Jordan algebras, one also introduces a generalization of the notion of the *principal minors* of a matrix [Ch. VI].

The Riemannian structure [Ch. I and Ch. III, Section 5]. Let $G = G(\Omega)$ and let K be the isotropy subgroup of ε. Then K is maximal compact in G, and $\Omega \cong G/K$ is a Riemannian symmetric space (hence the terminology, "symmetric cone").

Because a symmetric cone is an example of a noncompact Riemannian symmetric space, one has available the general theory, due to Harish-Chandra and Helgason, of harmonic analysis on Riemannian symmetric spaces. However, because of the interplay of the group theory with the Jordan structure, for symmetric cones one can describe much of the harmonic analysis in surprisingly fine detail and without recourse to the structure of semi-simple Lie groups or the general theory of noncompact symmetric spaces.[5]

In particular, the rich geometric and algebraic structure of a symmetric cone supports noncommutative generalizations of many of the explicit constructs of classical

[5] For an important exception see footnote 9.

analysis, including such special functions as the gamma function, Bessel functions, hypergeometric functions, and orthogonal polynomials; Taylor and Laurent series; various integral kernels and integral transforms; and Hardy and Bergman spaces, all of which are treated in the book.

Symmetric domains [Ch. X]. Let Ω be an irreducible symmetric cone and V the ambient simple Euclidean Jordan algebra. Then the complex domain $T_\Omega = V + i\Omega$ in the complexification $V^{\mathbb{C}} = V + iV$ of V is called the *tube domain* over Ω . In the classical case in which $V = V_1 = \mathrm{Sym}(1, \mathbb{R})$ is just the real line, Ω is the interval $(0, \infty)$, and $T_\Omega = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ is the ordinary upper half-plane H in $\mathbb{C}$, the 2×2 special linear group $\mathrm{SL}(2, \mathbb{R})$ acts transitively by linear fractional transformations on H, the special orthogonal subgroup $\mathrm{SO}(2)$ is the isotropy group of the point $i \in H$, and $H \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ has the structure of a Riemannian symmetric space. The general case gives rise to the same kind of structure. For if Ω is any symmetric cone, then the group $\mathcal{G} = \mathcal{G}(T_\Omega)$ of biholomorphic automorphisms of T_Ω acts transitively on T_Ω, the isotropy group $\mathcal{K}$ of the point $i\varepsilon \in T_\Omega$ is maximal compact in $\mathcal{G}$, and $T_\Omega \cong \mathcal{G}/\mathcal{K}$ is a Riemannian symmetric space. For this reason, T_Ω is referred to as a *symmetric tube domain*.[6] As in the classical case in which the Cayley transform maps the half-plane H biholomorphically onto the unit disk $\{w \in \mathbb{C} : |\, w\,| < 1\}$, in general there exist a bounded complex domain D in $V^{\mathbb{C}}$, called the *generalized unit disk* in $V^{\mathbb{C}}$, and a *generalized Cayley transform* that maps the tube domain $T_\Omega = V + i\Omega$ biholomorphically onto D.

When a Riemannian symmetric space has complex structure, as in the case of T_Ω, it is called a *Hermitian* symmetric space. Thus, D is a bounded realization and T_Ω an unbounded realization of the Hermitian symmetric space $\mathcal{G}/\mathcal{K}$, and both T_Ω and D are referred to as *symmetric domains*. In general, by realizing a Hermitian symmetric space as a symmetric domain in this way, one formulates a Jordan algebraic alternative to the more traditional Lie theoretic development.[7] The complex analysis developed in the book takes place primarily on $V^{\mathbb{C}}$ itself or a symmetric domain in $V^{\mathbb{C}}$.

The generalized gamma function Γ_Ω [Ch. VII]. One need only contemplate the significance of the classical gamma function in such subjects as complex analysis, classical harmonic analysis, the theory of special functions, probability and statistics, and analytic number theory to understand why one would want to generalize this function to the setting of a symmetric cone. For motivation, consider

[6]In analogy to the classical case, one also refers to T_Ω as the *generalized upper half-plane in* $V^{\mathbb{C}}$. For example, when $V = \mathrm{Sym}(m, \mathbb{R})$, the domain T_Ω arises in number theory and is commonly referred to as the *Siegel upper half-plane*.

[7]In general a Hermitian symmetric space can always be realized as a bounded complex domain in $\mathbb{C}^k$ for some k, but not all Hermitian symmetric spaces can be realized as tube domains. Those that can be realized in this way are said to be of *tube type*. For Hermitian symmetric spaces that are not of tube type, one can introduce an algebraic notion more general than a Jordan algebra, called a *Jordan triple system*, in terms of which one obtains an unbounded realization more complicated than a half-space, called a *Siegel domain of type II*. However, since only domains of tube type are treated in this book, neither of these more complicated constructs are needed. [The concept of a Jordan triple system is outlined in the notes to Chapter X.]

the classical gamma function, defined as the Euler integral

$$\Gamma(\lambda) = \int_0^\infty e^{-x} x^{\lambda-1} dx, \tag{6}$$

and introduce a Jordan algebra perspective by viewing the real line as $V = \mathrm{Sym}(1, \mathbb{R})$, the half-line $(0, \infty)$ as $\Omega = \mathrm{Pos}(1, \mathbb{R})$, and $\Omega \cong G/K$ where $G = \mathrm{GL}_+(1, \mathbb{R})$ is the multiplicative group of positive real numbers and K is the trivial subgroup of G consisting of the number 1 alone. Since the identity $\varepsilon \in V$ is just the number 1 and $d_*x = x^{-1}dx$ is G-invariant measure on Ω, we can rewrite (6) as

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-\langle x \,|\, \varepsilon \rangle} \Delta_\lambda(x) d_*x \tag{7}$$

where $\Delta_\lambda(x) = x^\lambda$ is the usual power function. Now, let Ω be any irreducible symmetric cone, denote its dimension by n and its rank by r, and reinterpret the symbols in formula (7) as follows: $\lambda = (\lambda_1, \dots, \lambda_r)$ is an r-tuple of complex numbers; for each such λ, the function Δ_λ is a generalization to Ω of the ordinary power function, called the *generalized power function* for Ω, defined explicitly in terms of the determinant and principal minors of x; and $d_*x = (\det x)^{-n/r}dx$ is G-invariant measure on Ω . Then the integral in (7) converges absolutely when $\mathrm{Re}\,\lambda_j$ is sufficiently large for each j, and (7) defines the *(generalized) gamma function* Γ_Ω for the cone Ω . For any irreducible cone Ω the generalized gamma function can be evaluated explicitly in terms of the classical gamma function; for example, when $\Omega = \mathrm{Pos}(m, \mathbb{R})$,

$$\Gamma_\Omega(\lambda) = (2\pi)^{m(m-1)/2} \prod_{j=1}^{m} \Gamma(\lambda_j - \tfrac{1}{2}(j-1)). \tag{8}$$

By means of formulas (7) and (8) one can develop the fine structure of the generalized gamma function, including generalizations of many of the identities (such as the functional equation) for the classical gamma function, and one can introduce analogues of well-known functions derived from the gamma function. In particular, one can define the *generalized beta function* and the *generalized Pochhammer symbol*, both of which play prominent roles in many areas of analysis on symmetric cones and symmetric domains.

About the book. This concludes our summary tour of symmetric cones and Jordan algebras, and we now turn to an outline of what one will find in the book. For purposes of review one can divide the book into two parts. The first part develops the algebra and geometry of symmetric cones, and the second part treats analysis on symmetric cones and symmetric domains.

The first half of the book. The first part of the book consists of Chapters I through VIII. There are only a few prerequisites for this portion of the book: a solid knowledge of linear algebra, some geometry and group theory, and an understanding of the classical gamma function. It would also be helpful to know some Lie theory, but even that is not necessary if one thinks in terms of the matrix examples. *In fact, my recommendation to anyone who reads the book is as follows: Each time a new abstract idea is introduced, take the time to work it out for the prototype example of real matrix space* $\mathrm{Sym}(m, \mathbb{R})$, *or the analogous complex matrix space* $\mathrm{Her}(m, \mathbb{C})$.

Chapters I through VI contain a careful, systematic, and complete development of the theory of symmetric cones, Euclidean Jordan algebras, and their interplay. The reader who wants to begin with a summary more detailed than the one given above, but still brief, might want to look at the first two sections of [1].

The gamma function for a symmetric cone is developed in Chapter VII, which can be viewed as a bridge between the algebra, geometry, and structure theory of the preceding chapters and the noncommutative real and complex analysis to follow. This is a crucial chapter, for as the authors point out in the preface, "the gamma function of a symmetric cone ... plays a central role in this book and can be said to determine the general flavour of the subject." Also in Chapter VII, as an application of the gamma function to be used later in Chapter XIII to study what are called weighted Bergman spaces on symmetric domains, the authors develop the theory of Riesz integrals and Riesz distributions on a symmetric cone, the background for which includes some quite technical Fourier analysis on $\mathbb{R}^n$. A reader less interested in these two topics can omit them without impeding progress through the book.

Chapter VIII, which is completely algebraic, is a thorough treatment of *complex Jordan algebras*. In this chapter one will find a generalization of the Jordan form of a matrix, a proof of the equivalence of the concepts of *Euclidean* and *formally real* Jordan algebras, and the proof that any complex semi-simple Jordan algebra can be realized as the complexification of a Euclidean Jordan algebra. This chapter is ancillary to the complex analysis that follows in the next two chapters and can even be omitted in a first reading.

In my view, this first part of the book would be suitable for a graduate level course, for in developing the theory of symmetric cones and Euclidean Jordan algebras, one brings together a rich interplay of algebra, geometry, topology, and real and complex analysis. The interested student can then go on to study some of the later chapters of the book, and possibly other literature, and be well-positioned for research in important areas of mathematics that are still evolving. One shortcoming of the book as a textbook involves the end of chapter exercises, which are rich in interesting and challenging problems (a number of which are drawn from the literature) but are light on routine problems that build self-confidence and hone calculational skills.

The second half of the book. The second part of the book, consisting of the remaining chapters, IX through XVI, is an exposition of a number of areas of analysis on symmetric cones and symmetric domains that can be viewed as noncommutative generalizations of well-known aspects of classical analysis. Consequently, in contrast to the first part of the book, which has few prerequisites, the required background for the second part is more extensive and includes, in addition to familiarity with classical analysis in one variable and Fourier analysis in several variables, a knowledge of basic Lie theory and group representations, including at least the correspondence between finite-dimensional representations of a Lie group and those of its Lie algebra, as well as the Peter-Weyl theory by which one decomposes a unitary representation of a compact group into its irreducible constituents. For a first reading I recommend Chapters IX and X, Chapters XI and XII, and Chapter XIV.

Chapters IX and X: Here one will find a Jordan algebraic development of the structure of symmetric domains of tube type and the associated complex analysis.

Chapter IX is a review of some basic theory, such as reproducing kernels, the Bergman space, the Hardy space of a tube domain, and the Shilov boundary and Poisson kernel. Chapter X contains the detailed geometry of symmetric domains, including the explicit construction of the Cayley transform and the generalized unit disk D, the calculation of the Shilov boundary Σ of D, and the description of the automorphism groups $\mathcal{G}(T_\Omega)$, $\mathcal{G}(D)$ and $\mathcal{G}(\Sigma)$. Also in Chapter X, the constructs introduced in Chapter IX are calculated in explicit detail.

Chapters XI and XII: Chapter XI describes the decomposition of the polynomial algebra $P(V)$, or equivalently $P(V^{\mathbb{C}})$, under the action of the group $G = G(\Omega)$, and Chapter XII extends the results to infinite series expansions that generalize ordinary power series and Laurent series. The following is a brief and simplistic summary of the basic ideas.

Let V be a simple Euclidean Jordan algebra of rank r, and denote by $\mathbb{Z}_+^r$ the set of all r-tuples $m = (m_1 \dots m_r)$ of integers such that $m_1 \geq \cdots \geq m_r$. If $m_r \geq 0$, we write $m \geq 0$ and say that m is nonnegative. The G-irreducible subspaces of $P(V^{\mathbb{C}})$ are indexed by the nonnegative $m \in \mathbb{Z}_+^r$, and for each such m there exists a nonzero polynomial $\Phi_m \in P(V^{\mathbb{C}})$, unique up to constant multiples, such that Φ_m is K-invariant.[8] Suitably normalized, Φ_m is called a *spherical polynomial.* Then any K-invariant polynomial f on $V^{\mathbb{C}}$ can be written as

$$f(z) = \sum_m a_m \Phi_m(z) \tag{9}$$

where the coefficients a_m are complex numbers, and the sum runs through a finite subset of indices $m \geq 0$.

Next, allow the sum in (9) to be infinite and to run through all indices $m \geq 0$. Then the right side of (9) is called a *spherical series.* If the series converges for some nonzero point in $V^{\mathbb{C}}$, then the series converges on an open domain about 0, and the series defines a K-invariant holomorphic function on that domain. Furthermore, in analogy to ordinary Taylor series, the domain of convergence is characterized by a generalization of Abel's lemma, and the coefficients a_m are determined by derivatives of f at 0. Formula (9) is then referred to as the *spherical Taylor series* of f.

Laurent series are more complicated. When $m \in \mathbb{Z}_+^r$ but m is not nonnegative, one can define a K-invariant rational function Φ_m whose singularities lie on the complement of the set $\mathcal{I}$ of invertible elements in $V^{\mathbb{C}}$. These *rational spherical functions* Φ_m have properties analogous to those of the spherical polynomials, and they appear in the decomposition of $\mathrm{L}^2(\Sigma)$. If on the right side of (9) we allow the sum to range over the entire infinite set $\mathbb{Z}_+^r$, then the series on the right side is a *spherical Laurent series.* When the series converges on a suitable domain (contained in the complement of $\mathcal{I}$), the left side defines a holomorphic K-invariant function on the domain. These concepts are fully developed in Chapter XII, not only for K-invariant holomorphic functions, but for holomorphic functions in general.

Chapter XIV treats spherical harmonic analysis on a symmetric cone. Here one finds a description of the algebra of G-invariant differential operators, *spherical functions* (of which the functions Φ_m in Chapter XI are special cases), the *spherical Fourier transform* (also known as the *Harish-Chandra transform*), and the so-called *Harish-Chandra c-function*, which arises in the inversion of the spherical transform.

[8] A function f on $V^{\mathbb{C}}$ is K-invariant if $f(k \cdot z) = f(z)$ for all $k \in K$ and $z \in V^{\mathbb{C}}$.

As noted above, this theory is a special case of K-invariant harmonic analysis on a Riemannian symmetric space, but the additional Jordan algebra structure available when G/K is a symmetric cone makes much of the theory simpler and more explicit.[9] We remark that Section 2 of Chapter XIV is an application of G-invariant differential operators to the fine structure of weighted Bergman spaces on the generalized unit disk that are defined in Chapter XIII (see below). Section 2 can be omitted on a first reading.

The remaining chapters of the book, XIII, XV, and XVI, are perhaps more specialized. (We cannot say "more technical", because the analysis outlined above is already quite technical.)

Chapter XIII, together with Section 2 of Chapter XIV, treats the theory of certain weighted Bergman spaces on symmetric domains, one of which coincides with the Hardy space associated to the Shilov boundary and others of which are Hardy spaces associated to the other boundary orbits. These spaces play an important role in the infinite-dimensional representation theory of the group $\mathcal{G}$ of the symmetric domain.[10] Chapter XIII concludes with a study of the so-called *Hua differential equations* that relate to the Poisson transform of functions on the Shilov boundary.

Chapters XV and XVI introduce the study of K-invariant special functions on a symmetric cone and associated integral transforms. In Chapter XV one will find quite explicit generalizations of hypergeometric functions, J-Bessel functions, Hankel transforms, and Laguerre polynomials; and in Chapter XVI is a generalization of the Wishart distribution, more on Hankel transforms, generalized K-Bessel functions, and some zeta integrals.

It should be clear now that this book is an important and welcome addition to the mathematics literature. To close on a personal note, I learned a lot from reading this book, I learned more upon a second reading, and I could learn still more if I read it again.

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[9] Perhaps the most important theorem in spherical harmonic analysis on Ω is the inversion formula for the spherical transform, of which the Plancherel formula for $\mathrm{L}^2(\Omega)$ is a corollary. This is the one notable exception in which Harish-Chandra's theory of harmonic analysis on semi-simple Lie groups is required in the Jordan algebraic setting. The following will give the rough idea. If Ω has rank r, then the spherical functions ϕ_λ are indexed by $\lambda \in \mathbb{C}^r$. For a K-invariant function f on Ω the *spherical Fourier transform* $\hat{f}$ is defined by

$$\hat{f}(\lambda) = \int_{\Omega} f(x)\phi_\lambda(x^{-1})d_*x \tag{10}$$

whenever the integral converges. Then there exists a function $\lambda \mapsto c(\lambda)$ called the *Harish-Chandra c-function* and a constant c_o such that for sufficiently "nice" f the *inversion formula*

$$f(x) = c_o \int_{\mathbb{R}^r} \hat{f}(i\lambda)\phi_{i\lambda}(x)\frac{d\lambda}{|\,c(i\lambda)\,|^2} \tag{11}$$

is valid. The inversion formula is a profound result in the theory of harmonic analysis on a Riemannian symmetric space. Currently there is no proof specific to symmetric cones, and the authors refer the reader to the proof for Riemannian symmetric spaces in general [4, Ch. IV, Theorem 7.5]. Nonetheless, in the case of a symmetric cone, the c-function is given explicitly in terms of the generalized beta function of the cone.

[10] However, to fully develop the representation theory (the study of what are called *representations of $\mathcal{G}$ of holomorphic type*, or alternatively *highest weight representations of $\mathcal{G}$*) requires more general Bergman spaces, not of complex-valued functions, but rather vector-valued functions [2].

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