BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, Number 1, January 1998, Pages 87–89
S 0273-0979(98)00734-4

Linear functional equations. Operator approach, by Anatolij Antonevich, Birkhäuser-Verlag, 1996, viii + 179 pp., \$123.00, ISBN 3-7643-2931-9

The study of the shift operator comprises a large part of operator theory; there are treatises devoted to the subject. The book under review is concerned, more generally, with operators which are linear combinations of compositions of multiplication operators with shifts (or translations or composition operators). Such operators b thus take the form

$$(\dagger) \quad bu(x) = \sum_{k=1}^{m} a_k(x)u(\alpha_k(x))$$

where the function u lies in some space, such as $L_p(X)$ for some measure space $(X, \mathfrak{B}, \mu)$, or $C(X)$ for some compact Hausdorff space X, or a Sobolev space, or a space of distributions (in which case u is not a function). Various conditions are imposed on the multiplication functions a_k, and on the translations (or shifts) α_k. The principal theme is spectral properties of the operator b : Is there a reasonable description of the spectrum? When is b invertible (i.e., $0 \notin \mathrm{sp}(b)$)? In what cases is $\mathrm{sp}(b)$ circularly symmetric?

This book is neither a monograph nor a textbook. Despite its relative brevity it treats a variety of topics, yet with thematic unity. As he says in the preface, the author has addressed the book to a broad circle of mathematicians, including specialists in functional analysis, differential and integral equations, and boundary value problems. He further claims it is accessible to 'senior students'. And indeed, it is accessible to any graduate student who has a general background in functional analysis and operator theory in particular, some ergodic theory, theory of manifolds, and differential equations. In my department, that would make for a small audience. Yet the book is worth looking at—especially by an instructor planning to teach a graduate course in one of the subjects mentioned.

Let me give an example of the kind of result which appears early in the book (as Theorem 2.1); while the result may appear elsewhere in one of the standard reference works on operator theory or ergodic theory, I cannot recall having seen it.

Let $X = \mathbb{R} \bmod 1$ be the unit circle, a a continuous function on X, and $0 < h < 1$. Let the operator $b, bu(x) = a(x)u(x+h)$ act on either $C(X)$ or $L_p(X), 1 \leq p \leq \infty$. Set

$$\Sigma_m(a) = \{\lambda \in \mathbb{C} : \exists x \in X, \lambda^m = \Pi_{k=0}^{m-1} a(x+kh)\}$$

and

$$M(|a|) = \exp(\int_0^1 \ln|a(x)|\, dx), \text{ or } M(|a|) = 0 \text{ if the integral diverges.}$$

Theorem. *If the number h is rational, and $h = l/m$ as a reduced fraction, then $sp(b) = \Sigma_m(a)$.*

1991 *Mathematics Subject Classification.* Primary 46-01; Secondary 47-01, 28-01.

If h is irrational, then

$$\mathrm{sp}(b) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| = M(|a|), \text{ if } a(x) \neq 0 \ \forall x\} \\ \{\lambda \in \mathbb{C} : |\lambda| \leq M(|a|), \text{ if } \exists x_0, a(x_0) = 0\}. \end{cases}$$

Furthermore, the spectrum of b and the Fredholm spectrum coincide in both cases (i.e., $u \in C(X)$ or $u \in L_p(X)$).

The proof makes use of standard graduate measure theory (the Monotone Convergence Theorem, which is referred to as Fatou's theorem), the spectral radius formula ($r(b) = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}}$), the upper semicontinuity of the spectral radius, and an instance of the Ergodic Theorem (which the author refers to as Weyl's Theorem). In my view it is a delightful application of a number of basic results; yet this example makes clear it is probably not accessible at the beginning graduate level.

Variants of this theorem appear in spaces of smooth functions, as well as generalizations to arbitrary compact spaces X and continuous maps $\alpha : X \to X$. The second chapter contains a discussion of C*-dynamical systems. Let $\mathcal{A}$ be a C*-algebra faithfully represented on a Hilbert space $\mathcal{H}$, and $\mathcal{G}$ a group acting on the same Hilbert space $\mathcal{H}$, by means of the unitary representation $g \mapsto T_g$. Suppose that $a \in \mathcal{A} \mapsto T_g a T_g^{-1}$ is an automorphism of $\mathcal{A}$, $g \in \mathcal{G}$. Then one can form the C*-algebra $\mathcal{B}$ which is obtained as the norm completion of the set of elements

$$\sum_{finite} a_g T_g.$$

These C*-algebras obtained from dynamical systems are usually not C*-crossed products; indeed, the crossed product norm is obtained by taking a supremum over all covariant representations. If $\mathcal{A}_1$ is a second C*-algebra, faithfully represented on some Hilbert space $\mathcal{H}_1$, and $g \mapsto U_g$ a unitary representation of $\mathcal{G}$ on $\mathcal{H}_1$, one can form the analogous C*-algebra $\mathcal{B}_1$. In contrast to the situation with C*-crossed products, even if the C*-dynamical systems $(\mathcal{A}, T_g)$ and $(\mathcal{A}_1, U_g)$ are isomorphic, one cannot conclude that $\mathcal{B}, \mathcal{B}_1$ are isomorphic C*-algebras. The isomorphism question is a delicate one. A positive result is presented for a restricted class of C*-algebras—those of the form HOM E , where E is an N-dimensional complex vector bundle, and the group action is topologically free.

Returning to individual operators of the form (†), there are few results known for operators with more than two terms (i.e., $m > 2$). In some cases it is possible, starting with a general b as in (†), to use a reduction procedure to get a two-term $\tilde{b}$. An instance in which it is possible to say something for operators b comprising more than two terms arises by specializing the maps α_k to shifts. For example, suppose the vectors $h_1, \ldots, h_m \in \mathbb{R}^l$ are rationally independent and form the vertices of a convex polyhedron in $\mathbb{R}^l$. Let $a_k \in L^\infty(\mathbb{R}^l)$, and assume that the limits $\lim_{x\to\infty} a_k(x)$ exist for $1 \leq k \leq m$. The operator b of form (†) with $\alpha_k(x) = x + h_k$, $1 \leq k \leq m$, as an operator on $L_p(\mathbb{R}^l), 1 < p < \infty$, is invertible if and only if the absolute value of one of the coefficient functions at ∞ dominates the sum of the others at ∞, and that function is bounded away from 0 outside a set of measure zero.

The preceding is an indication of the breadth of topics covered; the examples hardly form a complete list. Nothing has been said about the final chapter, which takes up questions concerning boundary value problems. The list of references contains some 258 items, and except for the last 50, these are primarily Russian.

The author and his colleagues have made substantial contributions to the material included. The book is well written with very few misprints. As it covers parts of large areas of mathematics, it is hard to compare with other books. I would say it would be of interest to a variety of specialities, including ergodic theory and control theory, but particularly to those working in operator theory.

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