Stochastic analysis, by Paul Malliavin, Springer, 1997, 343+xi pp., $125.00, ISBN
3-540-57024-1

1. Introduction

Suppose that $(\mathbb{W}, \mathcal{A}, P)$ is a Gaussian measure space (defined below), $F : \mathbb{W} \to \mathbb{R}^n$ is a measurable function, λ_n is Lebesgue measure on $\mathbb{R}^n$, and $P_F := P \circ F^{-1}$ is the image of P under F. We will refer to $F : \mathbb{W} \to \mathbb{R}^n$ as a Wiener functional and call P_F the law of F. The central theme in Malliavin's book is the following question.

Central Question: When is P_F absolutely continuous relative to λ_n with a density $\rho_F := dP_F/d\lambda$ which is smooth?

Before continuing, it is necessary to make precise the notion of a Gaussian measure space. We will then be in a position to motivate the central theme by giving some examples of interesting Wiener functionals.

2. Gaussian measures and Wiener functionals

For simplicity and clarity we will describe Gaussian measures only in the context of Gross' abstract Wiener space setting; see [12], [15]. Let $\mathbb{W}$ be a real separable Banach space equipped with its Borel σ –algebra, $\mathcal{A}$.

Definition 2.1. A measure P on $(\mathbb{W}, \mathcal{A})$ is said to be Gaussian (with mean zero) provided that there exists an inner product[1] $q : \mathbb{W}^* \times \mathbb{W}^* \to \mathbb{R}$ such that for all $\ell \in \mathbb{W}^*$,

$$\int_{\mathbb{W}} e^{i\ell(x)} dP(x) = \exp(-\frac{1}{2} q(\ell, \ell)).$$

(The inner product q may be shown to be continuous on $\mathbb{W}^*$.)

Gaussian measures play a central role in probability theory owing to the central limit theorem. They also have been important in physics and mathematical physics, since Gaussian measure spaces are intimately related to so-called free quantum field theories. See, for example, Segal [24], [25] and Glimm and Jaffe [11] and the references therein.

Given a Gaussian measure space $(\mathbb{W}, \mathcal{A}, P)$, for $\ell \in \mathbb{W}^*$ let $h_\ell \in \mathbb{W}$ be given by the Bochner integral, $h_\ell := \int_{\mathbb{W}} \ell(\omega)\omega dP(\omega)$. The mapping $\ell \in \mathbb{W}^* \to h_\ell \in \mathbb{W}$ is linear, injective, has dense range in $\mathbb{W}$, and satisfies $\|h_\ell\|_{\mathbb{W}} \leq (\text{const.})\sqrt{q(\ell, \ell)}$. for all $\ell \in \mathbb{W}^*$. The *reproducing kernel Hilbert* space $H \subset \mathbb{W}$ associated to $(\mathbb{W}, \mathcal{A}, P)$ is defined as

$$H := \{h = \lim_{n \to \infty} h_{\ell_n} \in \mathbb{W} \text{ where } \{\ell_n\}_{n=1}^{\infty} \subset \mathbb{W}^* \text{ is a } q\text{-Cauchy sequence}\}.$$

The inner product on H is defined by $(h, k) = \lim_{n \to \infty} q(\ell_n, \alpha_n)$, where $\{\ell_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are q-Cauchy sequences such that $h = \lim_{n \to \infty} h_{\ell_n}$ and $k = \lim_{n \to \infty} h_{\alpha_n}$ respectively.

---

[1]The condition that q be *positive* definite is included for simplicity. It insures that the support of P is $\mathbb{W}$.

Fact 2.2. *Although H is always dense in $\mathbb{W}$, H is equal to $\mathbb{W}$ iff* $\dim \mathbb{W} < \infty$.

Let us give two important examples of Gaussian measures.

Example 2.3. Suppose that $\mathbb{W}$ is finite dimensional and P is Gaussian with associated inner product q on $\mathbb{W}^*$. In this case $H = \mathbb{W}$ and $(\cdot, \cdot)_H$ is the dual inner product to q on $H = \mathbb{W}$. The measure P is given by the explicit formula:

$$dP(\omega) = (2\pi)^{-\dim(\mathbb{W})/2} \exp(-\frac{1}{2}(\omega, \omega)_H)\, d\omega,$$

where $d\omega$ denotes Lebesgue measure on $\mathbb{W}$ normalized to be one on unit cubes in $\mathbb{W} = H$.

Example 2.4 (Wiener Measure). Let $\mathbb{W}$ denote the space of continuous functions $\omega : [0, T] \to \mathbb{R}^d$ such that $\omega(0) = 0$. We make $\mathbb{W}$ into a Banach space by equipping it with the supremum norm. It is a celebrated theorem of N. Wiener [31] that there exists a unique Gaussian measure P on $(\mathbb{W}, \mathcal{A})$ such that

$$\int \exp\left(i \int_0^T \omega(t) \cdot d\mu(t)\right) dP(\omega) = \exp\left(-\frac{1}{2} \int_{[0,T]^2} \min(s,t) d\mu(s) \cdot d\mu(t)\right)$$

for all $\mu \in \mathbb{W}^*$ — where by the Riesz theorem $\mathbb{W}^*$ is being identified with $\mathbb{R}^d$ – valued measure on $[0, T]$. In this case H is called the *Cameron-Martin space* and consists of those $h \in \mathbb{W}$ which are absolutely continuous with the first derivative being square integrable. The inner product on H is given by $(h, k) = \int_0^T h'(t) \cdot k'(t)\, dt$.

Some key properties of Wiener measure are:

1. Wiener measure P represents a mathematical idealization of the probability distribution for the paths of particles diffusing in a homogeneous and isotropic environment. The next item makes this more precise by relating P to the heat equation on $\mathbb{R}^d$.
2. Given a continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$, one finds that the function $u(t, x) = \int_{\mathbb{W}} f(x + \omega(t)) dP(\omega)$ solves the heat equation $\partial u(t, x)/\partial t = \frac{1}{2}\Delta u(t, x)$ with $u(0, x) = f(x)$.
3. The set of paths in $\mathbb{W}$ which have a derivative at any one time $t \in [0, T]$ has P–measure zero and in particular $P(H) = 0$.
4. Let $\{V_i\}_{i=0}^d$ be a collection of smooth vector fields on $\mathbb{R}^n$ which are bounded along with all of their derivatives. (Here n is another integer.) Despite the previous item, it is possible using Itô's theory of stochastic calculus to give a meaning to the "solution" $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{W} \to \mathbb{R}^n$ to the "differential" equation

$$(2.1) \qquad d\gamma(t, x, \omega)/dt \quad = \quad V_0(\gamma(t, x, \omega)) + \sum_{i=1}^d V_i(\gamma(t, x, \omega))\dot{\omega}^i(t)$$

$$(2.2) \qquad\qquad \gamma(0, x, \omega) \quad = \quad x.$$

Such solutions produce quite complicated and interesting functions on $\mathbb{W}$. In particular, if $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous and bounded function as above, then

$$(2.3) \qquad\qquad u(t, x) = \int_{\mathbb{W}} f(\gamma(t, x, \omega)) dP(\omega)$$

is a solution to the generalized heat equation,

$$(2.4) \qquad\qquad \partial u(t,x)/\partial t = Lu(t,x) \text{ with } u(0,x) = f(x),$$

where L is the second order differential operator $L = \frac{1}{2}\sum_{i=1}^{d} V_i^2 + V_0$.

As a motivation to the central question stated above, consider the Wiener functional $F_{t,x}(\omega) := \gamma(t,x,\omega)$, where γ "solves" Eqs. (2.1) and (2.2) above. Supposing that $P_{F_{t,x}}$ has a smooth density $\rho_{t,x}$ relative to Lebesgue measure, one may write Eq. (2.3) as $u(t,x) = \int_{\mathbb{R}^n} f(y)\rho_{t,x}(y)dy$. Malliavin in his 1976 pioneering paper [19] gave a probabilistic proof of Hörmander's theorem in part by showing that $P_{F_{t,x}}$ has a smooth density. Recall that Hörmander's theorem asserts that any solution f to the equation $Lf = g$ is smooth when g is smooth provided the Hörmander condition is satisfied; i.e. the Lie algebra generated by the vector fields $\{V_i\}_{i=1}^{d}$ spans $\mathbb{R}^n$ for each point in $\mathbb{R}^n$. Malliavin's original paper was followed by an avalanche of papers carrying out and extending Malliavin's program, including the fundamental works of Stroock [28], [29], Kusuoka and Stroock [16], [17], and Bismut [1]. See also [2], [3], [4], [14], [22], [23], [26], [27], [30], the book under review, and the references therein.

3. MALLIAVIN'S IDEAS IN FINITE DIMENSIONS

To understand Malliavin's strategy for showing that a Wiener functional F has a smooth density, it is best to first consider the case where $\dim \mathbb{W} = N < \infty$.

Theorem 3.1. *Assume that $(\mathbb{W}, \mathcal{A}, P)$ is a finite dimensional Gaussian measure space in which case $H = \mathbb{W}$. Let $F : \mathbb{W} \to \mathbb{R}^n$ be a function and assume:*

1. *F is smooth and F and all of its partial derivatives are in $L^{\infty-} := \cap_{1 \le p < \infty} L^p(\mathbb{W}, P)$.*
2. *F is a submersion or equivalently assume the "Malliavin" matrix $M(\omega) := F'(\omega)F'(\omega)^*$ is invertible for all $\omega \in \mathbb{W}$.*
3. *Assume further that $\Delta(\omega) := \det(F'(\omega)F'(\omega)^*)^{-1} \in L^{\infty-}$.*

Then the law (P_F) of F is absolutely continuous relative to Lebesgue measure and the density $\rho := dP_F/d\lambda_d$ is smooth.

Proof. For each $v \in \mathbb{R}^n$, define $\tilde{V}(\omega) = F'(\omega)^* M(\omega)^{-1} v$—a smooth vector field on $\mathbb{W}$ such that $F'(\omega)\tilde{V}(\omega) = v$. Explicit computations using the chain rule and Cramer's rule for computing $M(\omega)^{-1}$ show that $D^k\tilde{V}$ may be expressed as a polynomial in Δ and $D^\ell F$ for $\ell = 0,1,2\ldots,k$. In particular $D^k\tilde{V}$ is in $L^{\infty-}$. Suppose $f,g : \mathbb{W} \to \mathbb{R}$ are C^1 functions such that f,g and their first order derivatives are in $L^{\infty-}$. Then by a standard truncation argument and integration by parts, one shows that $\int_{\mathbb{W}} (\tilde{V}f)g\,dP = \int_{\mathbb{W}} f(\tilde{V}^*g)\,dP$, where $\tilde{V}^* = -\tilde{V} + \delta(\tilde{V})$ and $\delta(\tilde{V})(\omega) := -\mathrm{div}(\tilde{V})(\omega) + \tilde{V}(\omega) \cdot \omega$.

Suppose that $\phi \in C_c^\infty(\mathbb{R}^n)$ and $v_i \in \mathbb{R}^n$; then

$$
\begin{aligned}
\int_{\mathbb{R}^n} (\partial_{v_1}\partial_{v_2}\cdots\partial_{v_k}\phi)dP_F &= \int_{\mathbb{W}} (\partial_{v_1}\partial_{v_2}\cdots\partial_{v_k}\phi)(F(\omega))\,dP(\omega) \\
&= \int_{\mathbb{W}} (\tilde{V}_1\tilde{V}_2\cdots\tilde{V}_k(\phi\circ F))(\omega)\,dP(\omega) \\
&= \int_{\mathbb{W}} \phi(F(\omega))\cdot(\tilde{V}_k^*\tilde{V}_{k-1}^*\cdots\tilde{V}_1^*1)(\omega)\,dP(\omega),
\end{aligned}
$$

where $\partial_v F$ denotes the partial derivative of F along v. By the remarks in the above paragraph, $(\tilde{V}_k^* \tilde{V}_{k-1}^* \cdots \tilde{V}_1^* 1) \in L^{\infty-}$. Hence, we find there is a constant C independent of ϕ such that

$$\left| \int_{\mathbb{R}^n} (\partial_{v_1} \partial_{v_2} \cdots \partial_{v_k} \phi) dP_F \right| \leq C \sup |\phi|.$$

It now follows from Sobolev imbedding theorems or simple Fourier analysis that $P_F \ll \lambda_n$ and that $\rho := dP_F/d\lambda_n$ is a smooth function. $\qquad \square$

4. The infinite dimensional case

In order to carry out the above procedure when $\dim \mathbb{W} = \infty$, it is necessary to understand how to correctly formulate the hypothesis of Theorem 3.1. The first question one is faced with is, what is the notion of a smooth Wiener functional? Since $\mathbb{W}$ is a Banach space, one might assume that a function $F : \mathbb{W} \to \mathbb{R}$ should be called smooth if it is differentiable to all orders relative to the norm topology on $\mathbb{W}$. However this notion of smoothness is too restrictive. Indeed most interesting Wiener functionals including "solutions" to equations like (2.1) are not even continuous on $\mathbb{W}$, let alone smooth. Hence it is crucial to relax the notion of smoothness. The appropriate calculus on Wiener space was initiated by Cameron and Martin [6], [7], [8] and Cameron [5]. They proved the following two results; see Theorem 2, p. 387 of [6], and Theorem II, p. 919 of [5] respectively.

Theorem 4.1 (Cameron & Martin 1944). *Let* $(\mathbb{W}, \mathcal{A}, P)$ *be classical Wiener space as in Example 2.4, and for* $h \in \mathbb{W}$*, set* $T_h(\omega) = \omega + h$*. Suppose that* h *is also* C^1*; then* PT_h^{-1} *is absolutely continuous relative to* P*.*

This theorem was then extended by Maruyama [21] and Girsanov [10] to allow the same conclusion for $h \in H$. (It is now known, on general Gaussian probability spaces $(\mathbb{W}, \mathcal{A}, P)$, that $PT_h^{-1} \ll P$ iff $h \in H$.) From the Cameron and Martin theorem one may prove Cameron's integration by parts theorem.

Theorem 4.2 (Cameron 1951). *Let* $(\mathbb{W}, \mathcal{A}, P)$ *be classical Wiener space and* $H \subset \mathbb{W}$ *be the Cameron–Martin space as in Example 2.4. Let* $h \in H$ *and* $f, g \in L^{\infty-}(P)$ *such that* $\partial_h f := \frac{d}{d\varepsilon} f \circ T_{\varepsilon h}|_{\varepsilon=0}$ *and* $\partial_h g := \frac{d}{d\varepsilon} g \circ T_{\varepsilon h}|_{\varepsilon=0}$ *where the derivatives are supposed to exist[2] in* $L^p(P)$ *for all* $1 \leq p < \infty$*. Then*

$$\int_{\mathbb{W}} \partial_h f\, g \, dP = \int_{\mathbb{W}} f \partial_h^* g \, dP,$$

where $\partial_h^* g = -\partial_h g + z_h g$ *and* $z_h := L^2(P) - \lim_{n \to \infty} \ell_n$ *where* $h = \lim_{n \to \infty} h_{\ell_n} \in \mathbb{W}$ *as in the definition of* H*.*

Armed with Cameron's integration by parts theorem and its extensions (see Gross [12], [13]), one may reasonably define a function $F : \mathbb{W} \to \mathbb{R}$ to be H–smooth provided all orders of the H–differentials of F exist and belong to $L^{\infty-}(P)$. This definition is sufficiently weak so as to include a large class of interesting Wiener functionals, including solutions to stochastic differential equations related to Eq. (2.1). Once the notion of a smooth Wiener functional is understood, Theorem 3.1 may be formulated when $\dim \mathbb{W} = \infty$. Proving the theorem and verifying the

---

[2]The notion of derivative stated here is weaker than the notion given in [5]. Nevertheless Cameron's proof covers this case without any essential change.

hypothesis in examples when $\dim \mathbb{W} = \infty$ requires a substantial amount of work to which a sizable portion of the book is devoted.

It would be highly misleading to say that the book is solely devoted to the proof of Theorem 3.1 and its applications when $\dim \mathbb{W} = \infty$. The ideas introduced by Malliavin and others to understand the smoothness properties of Wiener functionals have led to a very deep understanding of the analytic properties of Gaussian probability spaces. These topics are covered in *Stochastic analysis* as well. For example in Chapter 4, Malliavin develops the notions of capacities on Wiener space and explores the fine structure of Wiener functionals. In Chapter 5, these notions are used to define finite codimensional "submanifolds" of Wiener space. The differential and geometric properties of these infinite dimensional manifolds are then studied. And in Chapter 11, Malliavin introduces the reader to various recent results pertaining to the differential geometric analysis of "Wiener" measure on the path spaces of Riemannian manifolds.

The reviewer highly recommends this book to probabilists and non-probabilists alike. Malliavin masterfully guides his reader through a fascinating area of probability and analysis which has been active for 50 plus years. Moreover, by the end of the book a reader so inclined would be well equipped to study and work on problems of current interest in the field.

References

[1] Jean-Michel Bismut, "Large Deviations and the Malliavin Calculus," Birkhauser, Boston/Basel/Stuttgart, 1984. MR **86f:**58150

[2] Denis R. Bell, "The Malliavin Calculus," (Pitman monographs and surveys in pure and applied mathematics; 34), Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987. MR **88m:**60155

[3] ——— , "Degenerate Stochastic Differential Equations and Hypoellipticity," (Pitman monographs and surveys in pure and applied mathematics; 79), Longman, Essex, England, 1995. CMP 98:01

[4] Nicolas Bouleau and Francis Hirsch, "Dirichlet Forms and Analysis on Wiener Space", (Studies in Mathematics 14), de Gruyter, Berlin-New York, 1991. MR **93e:**60107

[5] R. H. Cameron, The first variation of an indefinite Wiener integral, *Proc. A.M.S.,* **Vol 2**. (1951), 914 - 924. MR **13:**659b

[6] R. H. Cameron and W. T. Martin, Transformations of Wiener integrals under translations, *Annals of Math.*, **45**, No. 2 (1944), 386 -396. MR **6:**5f

[7] R. H. Cameron and W. T. Martin, Transformations of Wiener integrals under a general class of linear transformations, *Trans. Amer. Math. Soc.* **58**, (1945) 184 - 219. MR **7:**127c

[8] R. H. Cameron and W. T. Martin, The transformation of Wiener integrals by non-linear transformations, *Trans. Amer. Math. Soc.* **66** (1949), 253 - 283. MR **11:**116b

[9] B. K. Driver, Towards calculus and geometry on path spaces, *in* "Stochastic Analysis, Summer Research Institute on Stochastic Analysis," July 11-30, 1993, Cornell University, (Eds. M. Cranston and M. Pinsky), Proceedings of Symposia in Pure Mathematics, Vol. **57**, American Mathematical Society, Rhode Island, 1995, p. 405-422. MR **96e:**60097

[10] I. V. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Theory Probab. Appl. 5 (1960), 285 - 301. MR **24:**A2986

[11] James Glimm and Arthur Jaffe, "Quantum physics. A functional integral point of view." Second edition. Springer-Verlag, New York-Berlin, 1987. MR **89k:**81001

[12] L. Gross, Abstract Wiener Spaces, Proc. 5th. Berkeley Symposium Math. *Stat. Prob.* **2**, (1965), 31 - 42.

[13] L. Gross, Potential theory on Hilbert space, *J. Func. Anal.* **1**, 123 - 181, (1967). MR **37:**3331

[14] N. Ikeda and S. Watanabe, "Stochastic differential equations and diffusion processes," 2nd ed., North-Holland Publishing Co., Amsterdam/Oxford/New York, 1989. MR **90m:**60069

[15] H-H. Kuo, "Gaussian measures in Banach spaces," Lecture notes in Mathematics, **463**, Springer-Verlag, Berlin-New York, 1975. MR **57:**1628

[16] S. Kusuoka and D. Stroock, Applications of the Malliavin Calculus, Part I, Proc. Int. Symp. S.D.E. Kyoto, (1976) 271 - 306, North-Holland, Amsterdam-New York, 1984. MR **86k:**60100a

[17] _____ , Applications of the Malliavin Calculus, Part II, *J. Fac. Sci. Univ. Tokyo*, Sect IA, Math, **Vol. 32** (1985), 1-76. MR **86k:**60100b

[18] P. Malliavin, Geometrie differentielle stochastique, Montreal: Presses de l' Universite de Montreal, 1978. MR **81d:**60077

[19] _____ , Stochastic calculus of variation and hypoelliptic operators, Proc. Int. Symp. S.D.E. Kyoto, (1976) 195 - 263, Wiley and Sons, New York 1978. MR **81f:**60083

[20] _____ , C^k-hypoellipticity with degeneracy, Stochastic Analysis, ed. by A. Friedman and M. Pinsky, 199-214, 321-340, Academic Press, New York, 1978. MR **80i:**58045a

[21] G. Maruyama, Notes on Wiener integrals, Kodai Math. Seminar Rep. 3 (1950), 41 -44. MR **12:**343d

[22] J. R. Norris, Simplified Malliavin calculus, Seminaire de Probabilites XX 1984/85 (ed. par J. Azema et M. Yor), Lect. Notes in Math., **1204**, 101-130, Springer-Verlag, Berlin, 1986. MR **89f:**60058

[23] D. Nualart, "The Malliavin calculus and related topics," in Probability and its Applications. Springer-Verlag, New York, 1995. MR **96k:**60130

[24] I. E. Segal, Tensor algebras over Hilbert spaces, I. *Trans. Amer. Math. Soc.* **81** (1956), 106-134. MR **17:**880d

[25] _____ , Distributions in Hilbert space and canonical systems of operators, *Trans. Amer. Math. Soc.* **88** (1958), 12-41. MR **21:**1545

[26] _____ , Absolute continuity of probability laws of Wiener functionals, *Proc. Japan Acad.,* **54-A**, 230-233 (1978). MR **81m:**60097

[27] _____ , Derivatives of Wiener functionals and absolute continuity of induced measures, J. *Math. Kyoto Univ.* **20-2**, 263-289 (1980). MR **83g:**60051

[28] D. W. Stroock, The Malliavin calculus and its application to second order parabolic differential operators, I, II, Math. Systems Theory 14, 25-65 and 141-171 (1981). MR **84d:**60092a, MR **84d:**60092b

[29] _____ , The Malliavin calculus, a functional analytic approach, *J. Funct. Anal.*, **44**, 212-257 (1981). MR **83h:**60076

[30] S. Watanabe, "Lectures on Stochastic Differential Equations and Malliavin Calculus, (Tata Institute of Fundamental Research: Lectures given at Indian Institute of Science, Bangalore), Springer-Verlag, Berlin-New York, 1984. MR **86b:**60113

[31] N. Wiener, Differential space, *J. Math. Phys.* **2**, 131-174 (1923).

BRUCE K. DRIVER
UNIVERSITY OF CALIFORNIA, SAN DIEGO
E-mail address: driver@euclid.ucsd.edu