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Harmonic analysis real-variable methods, orthogonality, and oscillatory integrals, by E. Stein, Princeton Univ. Press, Princeton, NJ, 1993, xiii+695 pp., $85.00, ISBN 0-691-03216-5

For almost three decades the two books [S2], [SW2] have been the basic reference for mathematicians working in harmonic analysis in $\mathbb{R}^n$. By these books hundreds of graduate students have been introduced to concepts like singular integrals, maximal functions, Littlewood-Paley theory, Fourier multipliers, Hardy spaces, to mention the main subjects.

The book under review is presented by the author as the third in the same series. So the general framework remains intentionally the same, and in large part the scope is to provide an updating of the same general subject. Still, whoever is familiar with the other books will find this one surprisingly different.

The reason is that harmonic analysis has seen an enormous growth in the past thirty years, and its implications in other fields of analysis have greatly diversified.

If the original work of Calderón and Zygmund was related to elliptic PDE's, later developments allowed applications of their theory to parabolic equations and to general hypoelliptic operators, and the more recent explosion of interest in the theory of oscillatory integrals and in problems involving curvature has much to do with hyperbolic equations.

Referring to another traditional aspect of harmonic analysis, as the borderline between real and complex analysis, there has been a considerable shift from one to several complex variables. Here again objects like hypoelliptic operators and oscillatory integrals play an important rôle; also non-commutative Lie group structures naturally appear.

If we add that this process has provided an occasion to revisit and clarify older concepts, one can easily understand that the book presents a considerable number of different facets and motives of interest for harmonic and non-harmonic analysts.

Introduction. As a preliminary introduction, let us recall some basic definitions and mention three classical problems that have been central in the growth of Euclidean harmonic analysis, particularly at the Zygmund school in Chicago: boundary behaviour of holomorphic and harmonic functions, L^p-regularity of solutions of elliptic PDE's, and convergence of Fourier series. The link among these apparently different problems is given by the concepts of singular integrals and maximal functions: in each case the relevant questions can be reduced to establishing boundedness on L^p (or weak L^1 estimate) for an appropriate maximal or singular integral operator.

The most elementary maximal function is the Hardy-Littlewood maximal function on $\mathbb{R}^n$

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|\,dy \;, \tag{1}$$

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where B_r is the ball of radius r centered at the origin, and $|B_r|$ is its Lebesgue measure. A basic fact in real analysis is that the differentiation formula

$$f(x) = \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r} f(x-y)\, dy \ , \tag{2}$$

valid a.e. for every locally integrable function f, follows from the fact that M is weak type $(1,1)$.

The Hilbert transform

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{1}{x-t} f(t)\, dt \tag{3}$$

is the most classical singular integral operator on the real line. Higher dimensional examples are the operators

$$Tf(x) = p.v. \int K(x-y) f(y)\, dy = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy \ , \tag{4}$$

where the convolution kernel $K(x)$ is homogeneous of degree $-n$, smooth away from the origin and has integral 0 on the unit sphere.

Maximal operators and singular integrals are so closely related that they constitute in fact two aspects of the same theory.

If f is a continuous function with compact support on the real line, its Poisson integral

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f(t)\, dt \tag{5}$$

defines a harmonic function on the upper half-plane $\mathbb{R}^2_+ = \{(x,y) : y > 0\}$, which solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ u = f & \text{on } \partial\mathbb{R}^2_+ \ , \end{cases} \tag{6}$$

with the additional condition that u vanishes at infinity.

If we apply (5) to a function $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, the resulting functon $u(x,y)$ is still harmonic in $\mathbb{R}^2_+$. In order to say that u is a solution of (6), we need to know, however, that

$$\lim_{y \to 0} u(x,y) = f(x) \qquad \text{a.e.} \tag{7}$$

But (7) is not so different from (2), and in fact it also follows from the properties of the Hardy-Littlewood maximal function.

Now, the function

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2 + y^2} f(t)\, dt \tag{8}$$

is also harmonic in $\mathbb{R}^2_+$, and it is called the *conjugate harmonic function* of u. It is characterized by the fact that $F = u + iv$ is a holomorphic function of $x + iy$ and by the condition that $v(\infty) = 0$.

A relevant question in the theory of Hardy spaces is whether the assumption that $f \in L^p(\mathbb{R})$ induces uniform L^p estimates on v in dx at all levels $y > 0$ (here we must require $p < \infty$ in order that the integral in (8) converges). It turns out that this is equivalent to requiring that the Hilbert transform be bounded on $L^p(\mathbb{R})$.

The fact that the Hilbert transform (as well as the more general singular integral operators in (4)) is bounded on L^p only for $1 < p < \infty$ is what makes the real variable theory of Hardy spaces (to which we will come back later) almost trivial for $1 < p < \infty$ and non-trivial for $p = 1$.

The Calderón-Zygmund theory of singular integral operators was mainly motivated by the regularity problem for solutions of linear elliptic PDE's. A typical question is: if D is an elliptic operator of order $2k$ with smooth coefficients, and $Du = f \in L^p$, can we say that the distributional derivatives of u of order $2k$ are locally in L^p?

In the simplest case, where D is the Laplacian, this question can be reduced to establishing the a-priori inequality $\|\partial^2_{ij}\varphi\|_p \leq C\|\Delta\varphi\|_p$ for test functions φ. Using the Fourier transform, one can see that

$$\partial^2_{ij}\varphi = R_i R_j(\Delta\varphi) ,$$

where R_i is the *Riesz transform*

$$R_i\varphi(x) = \int \frac{\xi_i}{|\xi|}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = p.v.\int c_n\frac{x_i - y_i}{|x-y|^{n+1}}f(y)\,dy \ . \tag{9}$$

Since these are nice singular integral operators, the answer is positive for $1 < p < \infty$. It is also interesting to observe that, as we shall see later, the Riesz transforms also intervene in the description of Hardy spaces in higher dimensions.

The most typical aspect of Fourier analysis is that one needs to control quantities that are sums, or integrals, of oscillating terms. The first hard question that arises in the study of the Fourier series of a 2π-periodic integrable function f is the almost everywhere convergence of its partial sums

$$S_n f(x) = \sum_{k=-n}^{n}\hat{f}(k)e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}f(x-t)\,dt \ .$$

As for the differentiation formula (2), a maximal function governs this convergence, namely

$$M^*f(x) = \sup_{n\in\mathbb{Z}}\left|p.v.\int_{-\pi}^{\pi}\frac{e^{int}}{t}f(x-t)\,dt\right| \ , \tag{10}$$

which has the additional feature of containing an oscillating factor.

The fact that the operator M^* is bounded on L^p for $1 < p < \infty$ [Ca2], [Hu] implies that the Fourier series of a function f converges to f as soon as $f \in L^p$ for some $p > 1$ (on the other hand, Kolmogorov's example shows that this is not true in general if f is only in L^1).

The classical examples that we have illustrated exhibit some of the main features of most problems in Euclidean harmonic analysis. As Stein says in his prologue, the three notions of maximal functions, singular integrals, and oscillatory integrals are fundamental, and the story of this field of mathematics can be seen as the story of the understanding of their interrelations and of their applicability to a large variety of problems.

The content of the book. Any attempt to give a presentation of Stein's book that is both synthetic and complete would be too ambitious. So we have chosen to focus on some of its main themes, at the price of many omissions and simplifications.

In doing so, we will not follow the author chapter by chapter; we will, however, indicate where the various subjects can be found in the book.

1. Real variable theory of Hardy spaces. The favorite definition of Hardy spaces for harmonic analysts has gone through subsequent modifications. According to the original definition, the Hardy space H^p (with $0 < p < \infty$) on the upper half-plane $\mathbb{R}^2_+ = \{x + iy : y > 0\}$ consists of the holomorphic functions such that

$$\|F\|_{H^p} = \sup_{y>0} \|F(\cdot, y)\|_{L^p(\mathbb{R})} < \infty \ . \tag{11}$$

This notion was extended by Stein and Weiss to higher dimensions [SW1] by replacing the notion of holomorphic function with that of *Riesz system*.

A Riesz system on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x_1, \dots, x_{n+1}) : x_{n+1} > 0\}$ is a vector-valued function $F = (f_1, \dots, f_{n+1})$ characterized by the generalized Cauchy-Riemann equations

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \text{for all } i \neq j, \quad \sum_{j=1}^{n+1} \frac{\partial f_j}{\partial x_j} = 0$$

(to be precise, these conditions describe the first-order Riesz systems; we disregard here higher-order Riesz systems, which are needed to define H^p when $p \leq \frac{n-2}{n-1}$). The defining condition (11) for H^p is then replaced by

$$\|F\|_{H^p} = \sup_{x_{n+1}>0} \|F(\cdot, x_{n+1})\|_{L^p(\mathbb{R}^n)} < \infty \ . \tag{11'}$$

Another step toward a real-variable theory of Hardy spaces consisted in putting the emphasis on the boundary values of H^p functions (or systems), rather than on the systems themselves.

In dimension one, let f be a distribution on the real line that grows slowly enough at infinity, so that its Poisson integral $u(x, y)$ and its conjugate function $v(x, y)$ are well-defined by (5) and (8) respectively. One says that $f \in H^p(\mathbb{R})$ if the holomorphic function $F = u + iv$ is in the "holomorphic" H^p space; i.e. if it satisfies (11). Observe that, from our previous remarks, $H^p = L^p$ for $1 < p < \infty$.

Similarly, in higher dimensions (and restricting to $p > \frac{n-2}{n-1}$ for simplicity), one says that a distribution f on $\mathbb{R}^n$ is in $H^p(\mathbb{R}^n)$ if its Poisson integral $u(x_1, \dots, x_{n+1})$ is the $(n+1)$-th component of a Riesz system $F = (v_1, \dots, v_n, u)$ on $\mathbb{R}^{n+1}_+$ belonging to the H^p space of Stein and Weiss. Again, $H^p = L^p$ for $1 < p < \infty$.

Two important results from the early '70s gave a big impulse to the theory of Hardy spaces: the characterization of $H^p(\mathbb{R}^n)$ in terms of maximal functions, and the realization of $BMO(\mathbb{R}^n)$ as the dual space of $H^1(\mathbb{R}^n)$ [FS].

The maximal functions to be considered here are constructed from smooth approximate identities. Let φ be a Schwartz function on $\mathbb{R}^n$ such that $\int \varphi = 1$, and set $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$. The corresponding maximal function is

$$M_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)| \ . \tag{12}$$

An even larger maximal function is the grand-maximal function

$$\mathcal{M}_N f(x) = \sup_{\varphi \in \mathcal{B}_N} M_\varphi f(x) \ , \tag{13}$$

where $\mathcal{B}_N$ is the unit ball in $\mathcal{S}$ in one of the norms $\| \ \|_{(N)}$ defining the Schwartz topology.

Theorem [FS]**.** *The following are equivalent, for $0 < p < \infty$ and $f \in \mathcal{S}'$:*
(i) *$f \in H^p(\mathbb{R}^n)$;*
(ii) *$M_\varphi f \in L^p(\mathbb{R}^n)$ for some $\varphi \in \mathcal{S}$;*
(iii) *$\mathcal{M}_N f \in L^p(\mathbb{R}^n)$ for some $N > N(p)$.*

There are many other alternative characterizations of H^p, also obtained by Fefferman and Stein, and which involve the Poisson integral $u(x,t)$ of f on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t \in \mathbb{R}_+\}$. To state two of them, let $\Gamma(x) = \{(y,t) : |y-x| < t\}$ be the upward cone pointed at $(x,0)$. Then each of the conditions

$$(14)\qquad \begin{aligned} u^*(x) &= \sup_{(y,t)\in\Gamma(x)} |u(y,t)| \in L^p \ , \\ Sf(x) &= \left(\int_{\Gamma(x)} |\nabla u(y,t)|^2 t^{1-n}\, dy\, dt \right)^{1/2} \in L^p \ , \end{aligned}$$

is also equivalent to $f \in H^p$.

A considerable simplification in the use of Hardy spaces is given by atomic decompositions, introduced by Coifman in one dimension [Co] and extended to other settings, including $\mathbb{R}^n$, by Coifman, Weiss and some of their students [CW].

For $0 < p \le 1$, a p-atom is a function $a(x)$ supported on a ball B and such that
(i) $|a(x)| \le |B|^{-1/p}$,
(ii) $\int x^\alpha a(x)\, dx = 0$ for $|\alpha| \le n(\frac{1}{p} - 1)$.

Then H^p, with $0 < p \le 1$, consists of those functions (or distributions) that can be written as sums $\sum_j \lambda_j a_j$, where the a_j are p-atoms and $\sum_j |\lambda_j|^p < \infty$.

This material is contained in Chapter III, together with applications to H^p-boundedness of singular integral operators.

2. BMO **and Carleson measures.** A locally integrable function $f(x)$ on $\mathbb{R}^n$ has *bounded mean oscillation* if, calling f_B the mean value of f on a ball B, one has

$$(15)\qquad \|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty \ .$$

This notion was introduced by John and Nirenberg [JN]. $BMO(\mathbb{R}^n)$ is defined as the space of equivalence classes, modulo constants, of functions having finite BMO-norm (observe that constant functions have zero BMO-norm).

The importance of BMO in harmonic analysis is due to C. Fefferman's theorem [FS], stating that $BMO(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$.

The notion of Carleson measure goes back to [Ca1]. Given a ball $B = B(x_0, r)$ in $\mathbb{R}^n$, let $T(B)$ be the cone in $\mathbb{R}^{n+1}_+$ with basis B, $T(B) = \{(x,t) : |x - x_0| \le r - t\}$. Then a positive Borel measure μ on $\mathbb{R}^{n+1}_+$ is called a Carleson measure if the μ-measure of $T(B)$ is controlled by the n-dimensional Lebesgue measure of its basis, i.e. if

$$(16)\qquad \mu(T(B)) \le C|B| \ ,$$

with C independent of B.

If $u(x,t)$ is the Poisson integral of a function $f(x)$ on $\mathbb{R}^n$, then (16) is equivalent to saying that for every $f \in L^p$, $1 < p < \infty$,

$$\int_{\mathbb{R}^{n+1}_+} |u(x,t)|^p \, d\mu(x,t) \le C \int_{\mathbb{R}^n} |f(x)|^p \, dx \ . \tag{17}$$

The following theorem provides a relation between BMO-functions and Carleson measures which is "dual", in some sense, to the maximal characterization of $H^1(\mathbb{R}^n)$ [FS].

Theorem. *The following are equivalent:*

(i) $f \in BMO(\mathbb{R}^n)$*;*

(ii) *if* $\varphi \in \mathcal{S}$ *is radial and* $\int \varphi = 0$*, then the measure*

$$d\mu(x,t) = |f * \varphi_t(x)|^2 \frac{dx\,dt}{t}$$

on $\mathbb{R}^{n+1}_+$ *is a Carleson measure.*

Carleson measures are presented in Chapter II, in connection with the other notion of "tent space". Chapter IV deals with BMO and its duality with H^1.

3. Spaces of homogeneous type. A considerable part of the theory of maximal functions, singular integrals, Hardy spaces and BMO requires very mild assumptions on the underlying space, and therefore it can be generalized to a large variety of different contexts.

One can replace $\mathbb{R}^n$ by a space X endowed with

(i) a locally compact topology defined by a quasi-distance d (this notion differs from the ordinary notion of distance in that the triangular inequality is replaced by the more general condition $d(x,z) \le C\big(d(x,y) + d(y,z)\big)$ for some constant $C \ge 1$);

(ii) a positive Borel measure m satisfying the doubling condition

$$m\big(B(x,2r)\big) \le C' m\big(B(x,r)\big)$$

for some constant C'.

Such a triple (X,d,m) is called a space of homogeneous type. Even though special instances had appeared before in the literature, the general notion was introduced by Coifman and Weiss [CW].

Relevant homogeneous-type structures on $\mathbb{R}^n$ which are different from the standard Euclidean structure are obtained by replacing

(i) the Euclidean distance with a non-isotropic distance

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|^{1/\delta_i} \tag{18}$$

for given exponents $\delta_i > 0$;

(ii) the Lebesgue measure with a weighted measure $dm(x) = w(x)\,dx$, where w is an A_∞ weight (see below).

Non-isotropic distances were originally used by Fabes and Rivière [FR] to extend the Calderón-Zygmund theory to parabolic equations.

The first part of Chapter I presents the basic theory of Hardy-Littlewood maximal functions and singular integrals in this general context. The underlying topological space is assumed to be $\mathbb{R}^n$, but the same proofs apply to general spaces.

4. A_p weights. (See Chapter V.) A non-negative locally integrable function w on $\mathbb{R}^n$ is an A_p weight, for $1 \le p < \infty$, if it satisfies the inequality

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,dx\right)^{p/p'} \le A < \infty\ , \tag{19}$$

for all balls B. Observe that for $p = 1$ the A_1-condition is equivalent to the pointwise inequality $Mw(x) \le Cw(x)$ a.e., where M is the Hardy-Littlewood maximal function.

The A_p classes are ordered by inclusion: $A_{p_1} \subset A_{p_2}$ if $p_1 < p_2$, and the union of all the A_p is called A_∞. Weights $w \in A_\infty$ are characterized by a "reverse Hölder inequality" on balls: there exists a $q > 1$ such that

$$\left(\frac{1}{|B|}\int_B w(x)^q\,dx\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\ , \tag{20}$$

for all balls B.

The rôle played by A_p weights is indicated by the following theorem.

Theorem. [M], [CF], [HMW] *The following are equivalent for $1 < p < \infty$:*

(i) *$w \in A_p$;*
(ii) *the Hardy-Littlewood maximal operator is bounded on $L^p\big(w(x)\,dx\big)$;*
(iii) *the singular integral operators in* (4) *are bounded on $L^p\big(w(x)\,dx\big)$.*

A_p weights also appear in other areas of analysis, like potential theory on Lipschitz domains, quasi-conformal mappings and calculus of variations.

5. Singular integrals and pseudo-differential operators. Pseudo-differential calculus is a fundamental tool to construct parametrices (i.e. inverses modulo smoothing operators) of linear differential operators and to study regularity of solutions of PDE's. Hence pseudo-differential operators (ψdo's in brief) are closely related to Calderón-Zygmund singular integral operators.

Such relations are discussed in Chapter VI and in part of Chapter VII, where the main boundedness properties of ψdo's are also presented. The key problems concern ψdo's of order zero,

$$Tf(x) = \int a(x,\xi)e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,d\xi\ , \tag{21}$$

with symbol $a(x,\xi)$ in one of the symbol classes $S^0_{\rho,\delta}$ with $0 \le \rho, \delta \le 1$, defined by the condition

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}\ , \tag{22}$$

for all multiindices α, β.

If we set

$$K(x,y) = \int a(x,\xi)e^{2\pi i\xi\cdot(x-y)}\,d\xi\ , \tag{23}$$

then K is the integral kernel of T, i.e.

$$Tf(x) = \int K(x,y)f(y)\,dy \tag{24}$$

(these formulas must be interpreted in the sense of distributions).

If $a \in S^0_{1,\rho}$, then K is a *Calderón-Zygmund kernel*; this means that it is a smooth function away from the diagonal $x = y$ and

$$\left|\partial_x^\alpha \partial_y^\beta K(x,y)\right| \le C_{\alpha,\beta}|x-y|^{-n-|\alpha|-|\beta|} \tag{25}$$

for all multiindices α, β.

By itself, (25) does not imply any boundedness for T. However, once boundedness on L^2 is known, then the standard Calderón-Zygmund theory gives boundedness on L^p for $1 < p < \infty$. A localized version of boundedness on H^p, $0 < p \le 1$, and on BMO also follows.

As to boundedness on L^2, this is relatively easy to prove if a belongs to the "standard" class $S^0_{1,0}$, and it follows from the Calderón-Vaillancourt theorem for $a \in S^0_{1,\rho}$ with $\rho < 1$. However, it may fail for $\rho = 1$.

6. Almost orthogonality. This crucial principle, also known as Cotlar-Stein's lemma, can be formulated in the general context of linear operators on Hilbert spaces, and it is widely used in harmonic analysis to obtain L^2 estimates, e.g. for singular integral operators on Lie groups, for oscillatory integrals, etc. It is presented in Chapter VII, where it finds its first application to the proof of the Calderón-Vaillancourt theorem, stating that ψdo's with symbols in $S^0_{\rho,\rho}$, $\rho < 1$ are bounded on L^2.

A sequence $\{T_j\}$ of linear operator on a Hilbert space $\mathcal{H}$ is called *almost orthogonal* if there is a numerical sequence $\gamma(j) \in \ell^1(\mathbb{Z})$ such that

$$\|T_i^* T_j\| + \|T_i T_j^*\| \le \gamma(i-j)^2 \ . \tag{26}$$

Theorem. *If $\{T_j\}$ is an almost orthogonal sequence, the series $\sum_j T_j$ is strongly convergent to a bounded operator T with $\|T\| \le C\|\gamma\|_{\ell^1}$.*

7. L^2 boundedness of Calderón-Zygmund kernels. The remaining part of Chapter VII deals with the extra conditions on a Calderón-Zygmund kernel K, such as in (25), that must be imposed in order to have boundedness on L^2 of the associated operator T.

If $K(x,y) = k(x-y)$ is a convolution kernel, then it is well known that boundedness of T on L^2 is equivalent to the condition $\hat{k} \in L^\infty$. In the general case, where no such simple criterion is available, it turns out that the key point is to test the operator on functions with compact support, normalized in an appropriate way. Precisely, a *normalized bump function* (nbf) on the ball $B = B(x_0, R)$ is a smooth function φ supported on B such that $|\partial^\alpha \varphi| \le R^{-|\alpha|}$ if $|\alpha| \le N$, for some N.

Theorem. *Let K satisfy* (25)*. Then T is bounded on L^2 if and only if*

$$\|T\varphi\|_2 \le CR^{n/2} \ , \tag{28}$$

for every R and every nbf on any ball of radius R.

This is a slightly different, but equivalent, formulation of the so-called "$T(1)$ theorem" of David and Journé [DJ].

8. Kakeya-type sets. Kakeya sets are sets of measure zero in the plane which contain a segment of unit length in any direction. Related sets are the Besicovitch sets, constructed in detail in Chapter X. A Besicovitch set E_ε is the union of $N = N(\varepsilon)$ rectangles with sides 1 and $1/N$, arranged in such a way that: (a) they have so many overlappings that $|E_\varepsilon| < \varepsilon$; (b) if each rectangle is translated by two units in the direction of its longer side, the new rectangles are pairwise disjoint.

Sets of this kind exist, and they are the source of a number of negative results in harmonic analysis having to do with maximal functions or convolution operators.

Given a family $\mathcal{R}$ of (hyper-)rectangles in $\mathbb{R}^n$, $n \geq 2$, let

$$M_{\mathcal{R}} f(x) = \sup_{R \in \mathcal{R}} \frac{1}{|R|} \int_R |f(x-y)|\, dy \ . \tag{29}$$

When $\mathcal{R}$ consists of all cubes centered at the origin, then $M_{\mathcal{R}} f$ is essentially the Hardy-Littlewood maximal function. It is natural to expect that $M_{\mathcal{R}}$ may fail to be bounded if the family $\mathcal{R}$ is too large.

If $\mathcal{R}$ consists of all rectangles with sides parallel to the coordinate axes and centered at the origin, then $M_{\mathcal{R}}$ is still bounded on L^p for $1 < p \leq \infty$, but it is no longer of weak type $(1,1)$.

More interesting is the case where $\mathcal{R}$ is the family of all rectangles centered at the origin but with arbitrary orientations. Testing $M_{\mathcal{R}}$ on the characteristic functions of Besicovitch sets, one can see that it can only be bounded on L^∞.

A similar test on characteristic functions of Besicovitch sets is used to prove the following result, concerning the so-called "ball multiplier" [F2].

Theorem. *Let B be the unit ball in $\mathbb{R}^n$, $n \geq 2$. The operator*

$$Tf(x) = \int_B \hat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi = f * \hat{\chi}_B(x) \tag{30}$$

is bounded on L^p if and only if $p = 2$.

The roundedness of the ball plays a crucial rôle here. If B denotes a polyhedron instead of a ball, the operator in (30) is bounded on every L^p with $1 < p < \infty$.

An important consequence of this theorem concerns spherical summation of multiple Fourier series. Let

$$\sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{ik \cdot x}$$

be the (multiple) Fourier series of a function f which is 2π-periodic in each of its variables. The spherical sums are

$$S_N f(x) = \sum_{\sum k_j^2 \leq N^2} \hat{f}(k) e^{ik \cdot x} \ . \tag{31}$$

Then $f = \lim_N S_N f$ in the L^p-norm for every $f \in L^p$ if and only if $p = 2$ (in contrast, polyhedral sums converge in the L^p-norm when $1 < p < \infty$).

9. Oscillatory integrals. In many problems one needs estimates either of definite integrals

$$\int e^{i\varphi(x)} \psi(x)\, dx \ , \tag{32}$$

or of operator norms for

$$Tf(x) = \int e^{i\varphi(x,y)} \psi(x,y) f(y)\, dy \ , \tag{33}$$

which take into account the cancellations due to the oscillatory factor $e^{i\varphi}$. In order to emphasize the dependence on the rapidity of the oscillations, one often introduces a real parameter λ in the exponent to multiply the phase function φ. Then the question is to understand the decay in λ of such quantities as $\lambda \to \infty$.

The history of integrals of the form (32), which Stein calls "oscillatory integrals of the first kind," is very old but still alive [PS], [Sc], [V]. The most classical results, presented in Chapter VIII, are van der Corput's lemma and the asymptotic stationary phase estimates. The key fact is that the decay in λ of the integral

$$I(\lambda) = \int_a^b e^{i\lambda\varphi(x)}\,dx$$

is determined by the degeneracy of φ at its stationary points.

The main application of these estimates in Fourier analysis concerns the decay at infinity of Fourier transforms of surface measures. Considering the graph of a smooth function $x_n = \Phi(x')$, the Fourier transform of its surface measure is given by

$$\int e^{-2\pi i\left(\xi'\cdot x' + \xi_n \Phi(x')\right)}\left(1+|\nabla\Phi|^2\right)^{1/2} dx' \ . \tag{34}$$

If S now is a smooth compact hypersurface in $\mathbb{R}^n$ and μ_S is its surface measure, then $\hat{\mu}_S(\xi)$ is a finite sum of terms as in (34). The behaviour of the phase function in (34) at its stationary points (which depend on the choice of a direction in the ξ-space) ultimately reflects the curvature of S at its various points. So, if S is *of finite type* (i.e. if it has a finite order of contact with its tangent lines), then

$$\hat{\mu}_S(\xi) = O\big(|\xi|^{-\varepsilon}\big) \tag{35}$$

as $\xi \to \infty$ for some $\varepsilon > 0$. The optimal decay, with $\varepsilon = \frac{n-1}{2}$, is attained when the Gaussian curvature of S never vanishes.

10. Restriction of the Fourier transform and Bochner-Riesz means. It is well known that if $f \in L^1(\mathbb{R}^n)$, then its Fourier transform $\hat{f}(\xi)$ is defined for every ξ and is a continuous function of ξ. In contrast, the Fourier transform of a generic function in L^p, with $1 < p \le 2$, is only defined almost everywhere.

Nevertheless, it turns out that if S is a surface with some amount of curvature (e.g. of finite type) in the ξ-space, and if p is close enough to 1, then an a-priori estimate

$$\|\hat{f}_{|S}\|_{L^2(S)} \le C\|f\|_p \tag{36}$$

holds for $f \in \mathcal{S}$. Thus the operator $f \longmapsto \hat{f}_{|S}$ extends to all $f \in L^p$ by continuity. This surprisingly gives a knowledge of *restriction of* $\hat{f}$ *to* S, despite S having measure zero.

It is easy to see that no estimate like (36) can possibly hold if S is flat; hence the curvature of S plays a fundamental rôle. As shown in Chapter VIII, the fact that (36) holds for some range of $p > 1$ is a direct consequence of the decay estimate (35). However, the more refined analysis of the exact range of exponents p, q for which the estimate $\|\hat{f}_{|S}\|_{L^q(S)} \le C\|f\|_p$ holds, for a given surface S, requires different ideas, and it still presents open problems, even for spheres. Taking $q = 2$, the best possible value of p for hypersurfaces with non-vanishing Gaussian curvature is $p = \frac{2n+2}{n+3}$. That this is in fact the case is proved in Chapter IX, as a consequence of a general $L^p - L^q$ estimate for oscillatory integrals "of the second kind", i.e. as in (33).

A somewhat related problem concerns the *Bochner-Riesz means* of order $\delta \geq 0$ of a function f on $\mathbb{R}^n$, defined as

$$S_R^\delta f(x) = \int_{|\xi|<R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i \xi \cdot x} \, d\xi \ . \tag{37}$$

The Bochner-Riesz summability problem for Fourier integrals concerns the validity of the limit formula

$$f(x) = \lim_{R\to\infty} S_R^\delta f(x) \ , \tag{38}$$

in the L^p-norm or almost everywhere. There is an analogous summability problem for multiple Fourier series. The Bochner-Riesz means of a function f on the n-dimensional torus are

$$S_R^\delta f(x) = \sum_{\sum k_j^2 < R^2} \hat{f}(k) \left(1 - \frac{\sum k_j^2}{R^2}\right)^\delta e^{ik\cdot x} \ .$$

Convergence of Bochner-Riesz means in the L^p-norm, both for Fourier integrals and for Fourier series, is equivalent to boundedness in L^p of the single operator $S^\delta = S_1^\delta$ in (37).

If $\delta = 0$, this is the ball multiplier in (30) and it is bounded only for $p = 2$. For positive values of δ, S^δ is bounded for a non-trivial range of p, depending on δ, around $p = 2$. This range is known precisely in dimension 2, but its determination is still an open problem in higher dimensions.

11. Fourier integral operators. Part of Chapter IX is concerned with L^p boundedness of Fourier integral operators

$$Tf(x) = \int e^{2\pi i \Phi(x,\xi)} a(x,\xi) \hat{f}(\xi) \, d\xi \ , \tag{39}$$

where $a(x,\xi)$ is in the symbol class $S_{1,0}^m$ and the phase $\Phi(x,\xi)$ is homogeneous of degree 1 in ξ. The relevance of these results in the study of hyperbolic PDE's, Radon transforms and in other problems involving propagation of singularities is also explained.

If $\Phi(x,\xi) = x\cdot\xi$, then (39) becomes a pseudo-differential operator and its integral kernel $K(x,y)$ is singular on the diagonal $x = y$. This implies that the singular support of Tf is contained in the singular support of f. In the general case, the kernel $K(x,y)$ is singular on the set

$$\Sigma = \{(x,y) : y = \nabla_\xi \Phi(x,\xi) \text{ for some } \xi\} \ ,$$

which governs the propagation of singularities produced by the operator T.

A natural non-degeneracy assumption for Φ is that

$$\det\left(\frac{\partial^2\Phi}{\partial x_i \partial \xi_j}\right) \neq 0$$

on the support of a, which allows us to obtain the following result.

Theorem. [SSS] *Let T be a Fourier integral operator with non-degenerate phase and symbol $a \in S_{1,0}^{-m}$, with $0 \leq m < \frac{n-1}{2}$. Then T is bounded on L^p for $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{m}{n-1}$.*

The proof is quite lengthy, and it involves many notions and techniques that the reader has already found in the previous chapters. The result follows by complex interpolation from an L^2-estimate for $m = 0$ and an $H^1 - L^1$-estimate for $m = \frac{n-1}{2}$. The L^2-estimate is a consequence of the fact that, when $m = 0$, T^*T is a better operator than T itself (in fact it is essentially a ψdo). At the other extreme ($p = 1$), the atomic decomposition of H^1 is needed together with a clever dyadic decomposition of the operator to compensate for the fact that the operator does not have a Calderón-Zygmund kernel.

12. Averages over submanifolds and related maximal functions. The maximal operators that the reader has encountered so far involve averages of functions with respect to probability measures that are absolutely continuous with respect to Lebesgue measure. Of a different nature in this respect is therefore the *spherical maximal function*

$$Mf(x) = \sup_{r>0} \int_{S^{n-1}} |f(x - r\omega)|\, d\omega \ , \tag{40}$$

where $d\omega$ is the normalized surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Motivations for considering M and its boundedness properties on L^p spaces come, for instance, from the study of the wave equation.

It is quite obvious that no boundedness (except for $p = \infty$) can hold in dimension 1. In higher dimensions, a simple test performed on a function f that grows like $|x|^{-n+1}$ at the origin shows that M cannot be bounded on L^p if $p \leq \frac{n}{n-1}$. Hence the following result is optimal. It was proved by Stein [S1] in dimensions $n \geq 3$ and, more recently, by Bourgain [B] in dimension 2.

Theorem. *The sherical maximal operator on $\mathbb{R}^n$, $n \geq 2$, is bounded on L^p for $p > \frac{n}{n-1}$.*

As a consequence, when $p > \frac{n}{n-1}$, the spherical averages of an L^p function f,

$$(A_r f)(x) = \int_{S^{n-1}} f(x - r\omega)\, d\omega$$

(initially defined for a.e. x and a.e. $r > 0$), can be extended by continuity to every $r > 0$ for a.e. x, and

$$\lim_{r \to 0} (A_r f)(x) = f(x) \qquad \text{for a.e. } x \ .$$

Chapter XI contains a detailed treatment of these issues in dimensions $n \geq 3$ (the two-dimensional case requires a rather different approach). The proof emphasizes the geometric aspects that are needed to extend this result to more general surfaces.

In fact, one can consider families of hypersurfaces that vary from point to point. For each $x \in \mathbb{R}^n$ one assigns a hypersurface S_x, and the averages contributing to $Mf(x)$ are taken over all dilates of S_x about x itself.

That a certain amount of curvature is required follows from the following observation: no L^p estimate for $p < \infty$ can occur if all the hypersurfaces contain a flat portion oriented in a fixed direction (like replacing the unit sphere in (40) by the boundary of the unit cube).

The curvature condition that intervenes is called *rotational curvature*: if the S_x are implicitly defined as $S_x = \{y : \Phi(x,y) = 0\}$, one requires that the Monge-Ampère determinant

$$\det\begin{pmatrix} \Phi & \partial_y\Phi \\ \partial_x\Phi & \partial^2_{xy}\Phi \end{pmatrix}$$

does not vanish on any S_x.

This includes the situation where $S_x = S$ is a fixed hypersurface with non-zero Gaussian curvature and, at the other extreme, certain distributions of hyperplanes, provided that their orientation changes "rapidly enough" as x moves.

A different problem, also concerning averages over submanifolds, arises if, instead of scaling the surfaces S_x in the ambient space, one restricts the averages to small neighborhoods of a fixed base point in S_x. This problem is treated in the translation-invariant case, i.e. with $S_x = x + S_0$, but allowing the manifold S_0 to have arbitrary dimension $k \le n-1$.

If S_0 is parametrized by means of a smooth function $\gamma(t)$ with $t \in \mathbb{R}^k$, the maximal function we are talking about is

$$Mf(x) = \sup_{r<1} r^{-k}\int_{|t|<r} |f(x-\gamma(t)|\,dt\ .$$

The curvature assumptions on S_0 that suffice to prove that M is bounded on some L^p with $p < \infty$ are restricted to the base point $\gamma(0)$, and in fact they can be very mild; for instance, M is bounded on the full range of L^p, $1 < p \le \infty$, provided S_0 is of finite type at $\gamma(0)$. The "flat" case, i.e. with S_0 not of finite type at $\gamma(0)$, is also interesting and non-trivial. References about the flat case are given at the end of Chapter XI.

13. Singular integrals on homogeneous groups. Motivations to study singular integral operators on homogeneous Lie groups originally came from two different areas of analysis: complex analysis in several variables and representation theory of semisimple Lie groups.

A homogeneous Lie group has $\mathbb{R}^n$ as its underlying manifold and its multiplication law is compatible with a family of non-isotropic dilations

$$\delta\circ(x_1,\dots,x_n) = (\delta^{a_1}x_1,\dots,\delta^{a_n}x_n)\ ,\qquad \delta>0\ ,$$

in the sense that such dilations are automorphisms: $\delta\circ(x\cdot y) = (\delta\circ x)(\delta\circ y)$.

The simplest non-commutative example is the Heisenberg group $\mathbb{H}^n$, whose elements can be expressed as pairs $(z,t)\in\mathbb{C}^n\times\mathbb{R}$, with product

$$(z,t)\cdot(z',t') = \big(z+z', t+t'+2\mathrm{Im}\langle z,z'\rangle\big)\ ,$$

and dilations $\delta\circ(z,t) = (\delta z,\delta^2 t)$.

Two proofs are given of L^p-boundedness of singular integral operators of convolution type

$$Tf(x) = f*K(x) = p.v.\int f(y)K(y^{-1}x)\,dy\ ,$$

where K is a Calderón-Zygmund kernel adapted to the given dilations.

Each of these proofs exploits different ideas and leads to different generalizations. Since homogeneous groups have a natural structure of spaces of homogeneous type, the key fact to be proven is L^2-boundedness.

The first proof, to be found in Chapter XII, is given on the Heisenberg group. Taking the Fourier transform in the central variable t, matters are reduced to a family of oscillatory integrals of the second kind on $\mathbb{C}^n$, called *twisted convolution operators*, of the form

$$T_\lambda f(z) = f *_\lambda K_\lambda(z) = p.v. \int f(w) K_\lambda(z-w) e^{2\pi i \lambda \langle z,w \rangle} \, dw \ , \tag{41}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $K_\lambda(z)$ is the partial Fourier transform of $K(z,t)$ in the t variable evaluated at λ.

The analysis of T_λ in (41) leads, on one hand, to a more general treatment of oscillatory singular integrals with a bilinear phase and, on the other hand, to unravelling the connections between twisted convolution, the Weyl correspondence in the formalism of quantum mechanics, and the Weyl calculus of pseudo-differential operators.

The second proof, contained in Chapter XIII, is given on general homogeneous groups. It is basically the original proof given in Knapp and Stein [KS], which constitutes the first application of the almost orthogonality principle.

14. The $\bar{\partial}_b$-complex on the Heisenberg group. The problems in several complex variables alluded to above, and which motivate the study of certain operators on the Heisenberg group, are explained in Chapter XIII. A preliminary part is meant to introduce the general formalism for the $\bar{\partial}$-Neumann problem on general domains in several complex dimensions and to explain the crucial rôle played by the boundary conditions. These involve the boundary $\bar{\partial}_b$-complex and the Kohn Laplacian

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^* \tag{42}$$

on $(0,q)$-forms.

This part is necessarily very sketchy and rather hard to follow for the non-expert. However, it motivates the computations that come next, that is, the explicit and neat construction of these objects in the "model" case of the Siegel domain (or generalized upper half-space)

$$\mathcal{U}^n = \left\{ (z_1, \dots, z_{n+1}) : \mathrm{Im} z_{n+1} > \sum_{j=1}^n |z_j|^2 \right\} \tag{43}$$

in $\mathbb{C}^{n+1}$. The reader who will follow the author all the way to the Appendix at the end of Chapter XIII will find the explicit solution, on $\mathcal{U}^n$, of the problem stated at the beginning of the chapter.

The Heisenberg group enters into this picture because it can be naturally identified with the boundary of $\mathcal{U}^n$ in such a way that the relevant differential operators are invariant under left translations. Similarly, the integral operators that appear as their inverses are convolution operators on the Heisenberg group.

In particular the study of (42) reduces to the analysis of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_n$ and of the Folland-Stein operators

$$\mathcal{L}_\alpha = \mathcal{L} + i\alpha \partial / \partial t$$

for appropriate values of α. The sub-Laplacian is a left-invariant differential operator with a "sum of squares" structure, i.e.

$$\mathcal{L} = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2 + Y_j^2) \ ,$$

where the X_j and the Y_j are left-invariant vector fields generating the Heisenberg Lie algebra. Setting $z_j = x_j + iy_j$ for $j \leq n$, we have

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} \ , \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} \ .$$

Since we are in dimension $2n+1$, $\mathcal{L}$ is not elliptic. However, the missing tangent direction, corresponding to the derivative $\partial/\partial t$, arises from the commutator of each X_j with the corresponding Y_j. So, by a celebrated theorem of Hörmander [Ho], $\mathcal{L}$ is hypoelliptic.

As to the operators $\mathcal{L}_\alpha$, Folland and Stein [FS] proved the formula

$$\mathcal{L}_\alpha\left((|z|^2 - it)^{-\frac{n+\alpha}{2}}(|z|^2 + it)^{-\frac{n-\alpha}{2}}\right) = \frac{c_n}{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}\delta_0 \ . \tag{44}$$

This implies that if $\alpha \neq \pm(n+2k)$ for some $k \in \mathbb{N}$, then $\mathcal{L}_\alpha$ is hypoelliptic. In fact the constant on the right-hand side of (44) is different from zero, so that one easily derives from this formula a fundamental solution of $\mathcal{L}_\alpha$ which is smooth away from the origin.

On the other hand, if $\alpha = \pm(n+2k)$, the distribution u on the left-hand side of (44) is a non-smooth solution of the homogeneous equation $\mathcal{L}_\alpha u = 0$. This shows that $\mathcal{L}_\alpha$ is not hypoelliptic.

A specific section of Chapter XIII concerns the famous Lewy operator,

$$Z = \frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t} \ .$$

This is the first known example [L] of an unsolvable operator (in the sense that there are smooth data f such that the equation $Zu = f$ has no solution in any neighborhood of a given point).

One easily realizes that Z lies in the complexification of the Heisenberg Lie algebra, namely $Z = \frac{1}{2}(X - iY)$, and that $Z\bar{Z} = -\mathcal{L}_1$. So the analysis of the Lewy operator fits naturally into this picture.

Also, the conditions that have to be imposed on f so that the Lewy equation $Zu = f$ can be solved near a given point explicitly involve the realization of $\mathbb{H}_1$ as the boundary of $\mathcal{U}_1$: the necessary and sufficient condition is that the Cauchy-Szegö projection of f on $\mathcal{U}_1$ can be continued analytically through the given point of the boundary.

About the book. Despite the intrinsic difficulty of the subject and the enormous amount of material that is treated, Stein has been able to make this book very readable, choosing an organization that tends to highlight the key notions and ideas and to explain their interrelations.

As a consequence, some theorems are not presented in their sharpest form, nor in their widest context; some applications are indicated but not fully pursued, etc. All this helps the reader to get a clearer idea of the salient parts. When the proofs become involved, they are often divided into sections so that the right emphasis can be given to explain the ideas arising at the various stages.

Each chapter, however, contains a last section of "Further results" and sometimes an appendix. Here one finds much of the material that has been filtered from the main body of the book, together with the presentation of a large number of applications and related material. Each single note can be the starting point of a long tour, very informative and documented. Sometimes this kind of research (aided by a very efficient index and a list of cited authors) crosses a large part of the book transversally.

I think that all readers can find their own way to access the book, depending on their expertise and their familiarity with the field.

A large part of the book can very well be the subject of one or more advanced graduate courses, especially if integrated with preliminaries from the previous books [S2], [SW2]. I have experienced that myself very successfully. However, a few chapters (in particular in the second half of the book) are very dense, and a certain amount of assistance may be desirable for a graduate student.

The more expert harmonic analyst will find Stein's approach and his synthetic view of the different issues very illuminating. I have personally profited from the parts of the book I am not particularly familiar with by organizing sparse ideas collected by listening to lectures or conversing with colleagues (and I doubt that many people, apart from the author, would feel confident with the whole content of the book).

The book is also recommended to a larger class of analysts, specifically to those who work in several complex variables and in various areas of PDE.

From the presentation given above, one is tempted to view Stein's book as a *summa* of harmonic analysis (and consequently to make a personal list of what is missing or understated). On the contrary, the book has a strong personal touch, which is not surprising if we consider the leading rôle played by the author in the research activity in the field. Quoting the author: "... I cannot deny that this book is in part autobiographical: as the narrator of the story, I have chosen to recount those matters I know best by virtue of having first-hand knowledge of their unravelling."

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