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A survey of knot theory, by Akio Kawauchi, Birkhäuser-Verlag, Basel, Boston, and Berlin, 1996, xxi+420 pp., $89.50, ISBN 3-7643-5124-1

1. Basic knots

Knot theory is a subject with a deep cultural background and a short mathematical history. The background of knot theory is marked by the entire past of human invention and discovery in ropework and weaving. In all this time, while knots were used for practical work and weaving [5], counting [6], games [13] and magic [18], no topological mathematics of knots was developed until the end of the last century. Of course, one may point out that the subject of topology had its real origin in Euler's work in the eighteenth century [14], but modern geometry has specific mathematical roots in ancient geometry, and even Euler's topology has its roots in older theorems such as the classification of the regular solids.

I believe that the cultural reason why the topology of knots was unarticulated until recent times has much to do with the practical origins of the subject. Knots in use involve the full physicality of the rope, its frictional properties, its thickness and its strength. To grasp a good mathematical model of all of this is a major problem and one that is really just beginning to be tackled by mathematicians [38]. Quipu [6], an aboriginal Peruvian method of counting and recording information with knots, used only the locality of knots in sequence on a rope and the distinguishability of different forms of knots seen "from the outside" as it were.

One may discern an origin of knot theory in the wonderful formula discovered by Gauss [19] for the linking number of two space curves. The Gauss formula was motivated by the problem of the strength of magnetic flux through a coil of wire, and it gives a relationship between the differential geometry of the curves and their topology. Nevertheless, Gauss did not go on to investigate the self-entanglement of curves that is the basis of knot theory.

It was left for Lord Kelvin (William Thompson) in the nineteenth century (see [29] for excerpts from his work) to ask a question that started mathematicians on the road to knots. Kelvin had the idea that atoms could be three dimensional vortices in the ether, smoke rings in the fluidity of space. In that time, it was assumed that empty space was a fluidic plenum filled with an elusive substance, the ether, that vibrated and flowed and carried light, heat, sound and the newly discovered radio waves. This is such a natural hypothesis that it can seem a bit strange to realize that it has disappeared under the onslaught of relativity, logical positivism and Occam's Razor. In Kelvin's time one could look at the swirls in the wake of a ship or the smoke rings floating from a good cigar and imagine that these vortices taken by analogy into the ether were the very source of the material world. One could imagine space itself and the patternings of that continuum to be the source of matter.

Kelvin commissioned Tait, Kirkwood and Little to make a table of knots with the intent that this would become a table of atoms. The idea of knots as atoms

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did not prevail, but this first systematic table of knots became the beginning of a mathematical investigation of the subject.

During this same period topology was beginning to emerge from geometry, algebra and analysis in the hands of Riemann and later Poincaré with his discovery of the fundamental group.

The knot tables show many interesting patterns. They were designed to exhibit the knots as diagrams where a knot diagram is a projection of a knot to a curve in the plane (with transversal self intersections, called crossings). At each crossing of the curve with itself one indicates by a broken arc which curve segment underpasses the other – from the point of view of that projection. In the case of the early tables, the under- and overcrossings in a diagram are not indicated if the diagram is alternating (in an alternating diagram the crossings go in the pattern under, over, under, over, ... as one moves along the curve). Tait, Kirkwood and Little discovered that all the knots with less than eight crossings were alternating, and they found the first non-alternating knots with eight crossings. They did not have the means to prove that these knots were non-alternating in all projections, nor to show that some of the pairs of alternating knots produced by changing all crossings (unders to overs and overs to unders) in a given diagram were topologically distinct. One says that a diagram K^* obtained by switching all the crossings in this way from the diagram K is the *mirror image* of K. Tait noticed that different projections of topologically equivalent alternating knots could be obtained from one another by a large scale move called “flyping”. He conjectured that this was always the case for alternating knot diagrams. The conjecture was finally proved in full generality by Menasco and Thistlethwaite [33], but this is getting ahead of our story.

In general the enterprise of making the knot tables raised a large number of questions, conjectures and ideas about the mathematics of knots. The combinatorial topology of Euler predates the knot tables, but Poincaré’s fundamental group was happening at nearly the same time. With the fundamental group came a precise tool for studying a knot by studying the topology of its complement. That is, if one regards a knot k as an embedding of a circle in three dimensional space S^3 (we use the three sphere S^3 for the ambient space of the knot), then the fundamental group of the complementary space, $\pi_1(S^3 - k)$, is an invariant of the topological type of the knot (S^3, k). This idea combined in the hands of Dehn, Seifert, Reidemeister, Wirtinger and others to produce algorithms for presenting the fundamental group from a given diagram of a knot. By analyzing the fundamental group one could begin the mathematical classification of knots. It was Max Dehn who first proved that the trefoil (the simplest knot) and its mirror image are topologically distinct.

The study of three dimensional manifolds and the study of knots and links developed in parallel. For one thing, covering spaces of the knot complement and branched covering spaces of the three-sphere along the knot are ways to study both the manifolds and the corresponding subgroups of the fundamental group of the base manifold. In 1920 J. W. Alexander [1] proved that every compact three manifold is a branched covering of the three sphere along an appropriate link. At this point the theories of knots, links and three manifolds became inseparable. In 1928 these ideas came to a special fruition in the work of J. W. Alexander [3]. Alexander formulated a polynomial $\Delta_K(t)$ associated with an oriented knot or link K. Alexander’s polynomial had the property that it was (up to multiples of $\pm t^{\pm 1}$) an invariant of the topological type of the knot. The polynomial was very useful in distinguishing many knot types. It was, however, not capable of distinguishing a knot from its mirror image.

Alexander's polynomial was motivated by the fundamental group and covering spaces of the knot complement, but he presented the paper as a combinatorial algorithm to compute the knot polynomial from its diagram. A diagrammatic method had been formulated by that point by K. Reidemeister [36]. Reidemeister proved the remarkable theorem that if K and K' are diagrams for topologically equivalent knots (or links), then there exists a series of transformations of these diagrams based on three move types taking one diagram to the other (up to planar isotopy not affecting the structure of the diagram). Reidemeister's Theorem provided a combinatorial basis for the knot theory that applied directly to the original diagrams of Tait and Kirkwood and Little. Alexander took advantage of Reidemeister's Theorem to prove the invariance of his polynomial. Reidemeister wrote the first book on knot theory [36]. He based his approach on combinatorial topology and these Reidemeister moves.

During this same period Alexander [2] showed that every knot was topologically equivalent to a closed braid, and Emil Artin [4] developed the theory of braids as an algebraic structure. A little later Markov [32] proved the key theorem interrelating the study of knots, links and braids. Markov's Theorem made it possible, in principle, to study knot theory entirely in terms of braids. This approach later became highly developed [8].

While the Reidemeister moves provided a combinatorial basis for the Alexander polynomial, it was quickly recognized through the work of Seifert [39] that three dimensional techniques could illuminate its structure. Seifert showed how to calculate the Alexander polynomial by using an orientable surface whose boundary is the knot or link. Seifert gave an algorithm to produce such a surface from any link diagram, and he gave a formula for the Alexander polynomial in terms of linking numbers of curves on this surface. As a result, it became easy, given a polynomial with a certain symmetry of coefficients, to produce knots or links with that given polynomial as the Alexander polynomial. It became transparent how to produce non-trivial knots with Alexander polynomial equal to one.

Given any orientable surface F embedded in three-space, Seifert defines a bilinear (asymmetric) pairing on the first homology group of F by the formula

$$\Theta : H_1(F) \otimes H_1(F) \longrightarrow Z,$$

$$\Theta(a, b) = lk(a^*, b),$$

where $lk(x, y)$ denotes the linking number of x and y and a^* denotes the result of translating the cycle a into the complement of F by a small amount along the direction of the positive normal to the surface. The Seifert pairing Θ is an invariant of the embedding of the surface F in three-space. Seifert proved the

Theorem. $\Delta_K(t) \doteq Det(\Theta^T - t\Theta)$

Here $\doteq$ denotes equality up to sign and integral powers of t, and Θ^T denotes the transpose of Θ. The determinant is computed by using a matrix of Θ in some basis for the first homology group of F.

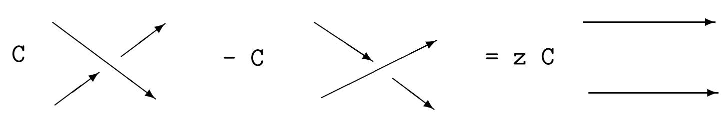
The Seifert pairing turns out to be one of the great devices of knot theory. It continues to play an important role to the present day.

The Alexander polynomial presented quite a puzzle to knot theorists, and it was not until the early 1950's that its relationship with the fundamental group of the knot complement was made completely clear in the work of Ralph Fox. Fox developed a non-commutative calculus (the Fox free differential calculus) related to the chains on a covering space, but formulated in pure algebraic terms. With the help of this calculus the Alexander polynomial was seen to be an invariant of the group of the knot. The polynomial can be extracted from any finite presentation of the knot group by pure algebra. A beautiful account of this theory can be found in the book by Crowell and Fox [12] and in the article by Fox entitled "A Quick Trip through Knot Theory" [16]. Another remarkable property of the Alexander polynomial comes through the work of Fox and Milnor [17]. They proved that a knot in the three-sphere that bounds a smooth disk in the four-dimensional ball (with boundary that three-sphere) has Alexander polynomial of the form $f(t)f(t^{-1})$ for some polynomial $f(t)$. This result marked the beginning of the study of *knot cobordism* and is a key fact in the interface between three and four dimensional topology.

John Milnor's discovery of exotic differentiable structures on higher dimensional spheres [34] propelled the focus of geometric topology in the 1960's and 1970's into higher dimensions. Higher dimensional knot theory is the study of embeddings of n-dimensional manifolds in $(n+2)$-dimensional manifolds. In all cases the fundamental group can still be used, along with other invariants from algebraic topology and surgery theory. In the case of embeddings in (homology) spheres, the Seifert pairing (appropriately generalized) is still available and very important [31]. With the examples of Breiskorn and the work of Milnor [35] the original exotic spheres were seen to be knots in high dimensions! These exotic knots can be constructed by branched coverings and generalizations of branched coverings in a pattern of geometric topology that goes right back to Alexander and Seifert (see [23], [24] and [28]).

The work of Casson and Gordon [9] began a return of interest in low dimensions. They gave examples of knots that did not bound disks in the four ball (non-slice knots) where this property was undetectable by the Alexander polynomial or by the Seifert pairing. These results are special to three dimensions. Another movement towards dimension three was the development of the Kirby calculus [30], [15] reformulating the classification of three manifolds in terms of moves on framed link diagrams representing the three manifolds via surgery on the corresponding links.

During this period of the 1970's there was another collection of results in knot theory that slowly came to be appreciated. This was the work of John H. Conway explained in concise form in his paper [10] and in lectures and conversation. His work told how to compute the Alexander polynomial (single and multi-variable versions) in terms of a recursive algorithm on the diagrams. In fact, Conway's method leads to a refinement of the Alexander polynomial that is often called the Conway (Alexander) polynomial. The one-variable version of the recursion relation for the Conway polynomial is shown below.



In symbols it reads

$$C_{K_+} - C_{K_-} = zC_{K_0}$$

where K_+, K_- and K_0 denote identical diagrams except at the site of one crossing that is (respectively) positive, negative or smoothed. One takes this basic exchange relation, plus the stipulation that the Conway polynomial is an invariant of topological knot type (exactly, not up to signs or powers of the variable) and normalization to one on the unknot. The Conway polynomial can then be computed entirely from the diagrams without recourse to determinants or any algebra other than high school algebra! Conway stated that his polynomial and the Alexander polynomial are related via the formula

$$C_K(z) \doteq \Delta_K(t)$$

for $z = t^{1/2} - t^{-1/2}$, leaving the proof of this puzzle to his readers or listeners! This approach to knots and links is often called *skein theory*.

Skein theory captured attention again around 1978 when John Conway gave talks on his polynomials in a number of places. This reviewer [25] and independently Cole Giller [20] found a model for the Conway polynomial using the Seifert pairing. Later, Hartley [21] wrote a paper showing how to model the polynomial via the Fox calculus. This reviewer found a combinatorial model [26] based on *state sums* that reformulated Alexander's original approach.

A state summation defines the invariant via a collection of *states* of the diagram, each state contributing a product of *vertex weights* to the sum over all the states. Rewriting the Alexander polynomial as a state sum was a prescient step, as shown by the events that followed! In 1984 Vaughan Jones [22], using the trace of a representation of the Artin braid group to a von Neumann algebra, produced a new (Laurent) polynomial invariant of knots and links, denoted $V_K(t)$, that outdid the Alexander polynomial in a number of ways. In particular the Jones polynomial could distinguish many knots from their mirror images (one says that the polynomial could detect the chirality of some knots). On top of this Jones showed that his polynomial also satisfied a Conway – type identity, namely

$$t^{-1}V_{K_+} - tV_{K_-} = (t^{1/2} - t^{-1/2})V_{K_0}.$$

Suddenly, the Conway method of skein calculation had two valid topological examples, and a bunch of topologists realized that the skein method had extraordinary potential. The first discovery was a direct generalization of the original Jones polynomial to an invariant $P_K(a, z)$ in two variables a and z such that

$$aP_{K_+} - a^{-1}P_{K_-} = zP_{K_0}.$$

P_K is called the *Homflypt polynomial* after the different people that discovered it and proved its properties. They are Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytcki and Trawzyk in the order of the acronym. The collaborative pairs are LM, FY and PT. In all cases except Ocneanu and Freyd-Yetter, this new polynomial had its properties analyzed by pure induction on the diagrams and the Reidemeister moves. (To avoid proliferation of references, we now refer the reader to the book under review for references to the source papers for the new polynomials.)

A few months later Millett and Ho found another new invariant $Q(z)$ (in one variable) on *unoriented* diagrams that satisfied the identity

$$Q_K + Q_{K'} = z(Q_{EK} + Q_{EK'})$$

where K and K' differ by the switch of one crossing and EK and EK' are diagrams that result from smoothing the crossing in the two ways that are possible without orientation. The Q polynomial was the first explicit appearance of an invariant that did not depend on the orientation of the link K. The Q-polynomial does not detect chirality.

The reviewer, in 1985, found a way to generalize the $Q-$polynomial to a two variable polynomial $F_K(a,z)$ that can distinguish many mirror images. This generalization involves using a representation of knot diagrams that respects twisting.

We need the concept of a *curl* in the diagram. A curl is a place in the knot diagram where an observer moving along the diagram goes through a crossing twice in quick succession with no intervening crossings before the repeat takes place. The curl has the appearance of a little loop in the rope.

One first defines a polynomial $L_K(a,z)$ that is invariant only under the second and third Reidemeister moves and so that

$$L_K + L_{K'} = z(L_{EK} + L_{EK'})$$

and

$$L_{K(+)} = aL_K,$$

$$L_{K(-)} = a^{-1}L_K$$

where $K(\pm)$ denotes a diagram with a single curl of positive or negative type, and K denotes the result of removing the curl from that diagram. Note that L has the same skein relation as Q. The difference between the two polynomials is in the way that they treat the twisting data corresponding to the curl. The fully invariant polynomial F_K is defined by the formula

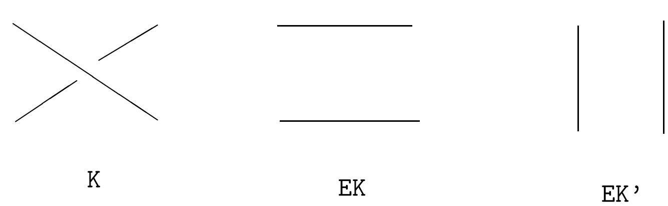
$$F_K(a,z) = a^{-w(K)}L_K(a,z)$$

where $w(K)$ denotes the sum of the signs of the crossings of the oriented link K.

In the wake of this generalization came a state summation model for a specialization of L_K [27]. In this model, denoted $< K >$, we have

$$< K > = A < EK > + A^{-1} < EK' >$$

where A is an independent variable making $< K > (A)$ a Laurent polynomial in A. Here EK denotes the result of smoothing a given crossing in the diagram K, and EK' denotes the result of smoothing the switch of that crossing. This relation is indicated below.



K EK EK'

In this case, one also has that

$$< OK > = d < K >$$

where O denotes an unknotted closed loop disjoint from the diagram K and $d = -A^2 - A^{-2}$. The bracket model can be construed as a sum over "states" S of the diagram K that are obtained by making a choice of smoothing at each crossing. Then

$$< K >= \sum_S < K|S > d^{||S||-1}$$

where $||S||$ denotes the number of simple closed curves in the state S and $< K|S >$ denotes the product of vertex weights (equal to A or A^{-1}) associated with the smoothed crossings in S. This state summation is very closely related to the partition function of the Potts model in statistical mechanics, and it provides a bridge between these two subjects. The bracket state model gives a direct construction of the original Jones polynomial. The formula is

$$V_K(t) = f_K(t^{-1/4})$$

where

$$f_K(A) = (-A^3)^{-w(K)} < K > (A).$$

The Laurent polynomial in A, $f_K(A)$, is an invariant of ambient isotopy for knots and links in three space. As the formula above states, the original Jones polynomial is a reparametrization of $f_K(A)$.

The bracket state model for the Jones polynomial was followed by a big influx of ideas from statistical mechanics applied to knot theory. In the bracket polynomial the combinatorial structure underlying the partition function is the knot or link diagram itself. In statistical mechanics it is customary to use a graph as the underlying combinatorial structure. It became apparent that one could mimic the assignment of vertex weights in statistical mechanics in the category of knot and link diagrams. In particular, one could import certain matrices, called "solutions to the Quantum Yang-Baxter Equation" to do the main job for making vertex weights in such a model. A given solution to the Quantum Yang-Baxter Equation would always provide a representation of the Artin braid group, and sometimes it can be augmented to provide an invariant of knots and links. At this stage it was noted by Jones, Turaev and others that this use of the Yang-Baxter equation provided a way to construct a series of specializations of all the known skein polynomials (as described above). Logically these specializations are sufficient to determine the existence of the skein polynomials. In this sense, all the skein polynomials are subsumed under the theory of these *quantum link invariants*.

The solutions to the quantum Yang-Baxter equation are closely related to the theory of *quantum groups* (non co-commutative Hopf algebras) which are in turn deeply related to the classical Lie algebras. As a result, the quantum groups became tied in with the knot theory, and a complex texture of algebra and topology arose.

These theories took a new turn in the late 1980's with Edward Witten's discovery [40] of a far-reaching relationship between knot invariants and quantum field theory. By interpreting knot invariants as certain functional integrals he showed how the invariants derived from quantum groups related to classical Lie algebras could be taken directly from those Lie algebras and their representations. He also showed how the quantum field theory context defined many invariants of three dimensional manifolds and how these invariants could be calculated (by a surgery description) through invariants of knots and links in the three dimensional sphere. At the same time, Reshetikhin and Turaev [37] discovered a method of defining invariants of three manifolds by using the algebraic structure of representations of quantum groups. It became clear that (to the extent that the Witten invariants are well-defined) the invariants of Witten and the invariants of Reshetikhin and Turaev coincide.

This development was a landmark in the evolution of a new theory of knots, links and invariants of three manifolds. Along with the quantum field theoretic viewpoint came a notion of *topological quantum field theory* [7] (forged by Witten and by Atiyah) that gave a framework into which many new results could fit. The perturbation expansion of Witten's integral promised more insights. It has delivered these in recent years in relation to other forms of three manifold invariants and in the formulation of Vassiliev invariants. We will say little about these in this review other than to point out that Vassiliev invariants provide a way to make a theory of knot and link invariants that is based directly on Lie algebra or generalized Lie algebra data. The Vassiliev invariants are closest in formalism to the perturbation series of Witten's integral.

So far we have given a sketch of the development of the theory of knots and links and its radical transformation in the last fifteen years. Now we turn to the contents of the book *A survey of knot theory* by Akio Kawauchi.

2. A survey of knot theory

The book by Kawauchi is not just a survey of knot theory. It is in fact a small but feisty encyclopedia of the subject. It achieves this aim in a compact space by accurate statements of theorems and examples and by eschewing the details of the proofs of many theorems. The book is endowed with an excellent index that this reviewer has tested on a number of occasions. It is actually possible to ask a specific technical question in knot theory, use this index and get significant information! Where the index fails, the table of contents often succeeds. For example, I just looked for the word "mutant" in the index. It is absent from the index. However, chapter heading 3.8 refers to this term, and turning to that section I find a lucid discussion of the meaning of the term and the fundamental example of the mutant pair of knots: the Conway knot and the Kinoshita-Terasaka knot. Each of these knots has Alexander polynomial equal to one. Furthermore, the text reminds the reader of the fact that the Kinoshita-Terasaka knot bounds a smooth disk in the four-ball (i.e. it is a slice knot), while it is an unsolved problem whether the same is true of the Conway knot.

This is what I like about this book. You start by looking up something in it, and it ends up by telling you about an interesting problem that rivets your thought. Another good feature of the book is an extended list of references to the literature arranged alphabetically by author. The book comes equipped with six appendices, the first five on background topics (for example, Appendix C is an account of the canonical decompositions of three-manifolds with clear statements of Dehn's Lemma, the Loop and Sphere theorems, properties of Haken manifolds and properties of hyperbolic three-manifolds). The sixth appendix is a new tabulation of the knots through 10 crossings with information about symmetry and skein polynomials. These appendices make the book particularly useful for researchers in three dimensional topology.

Now, turning to the book's overall organization, we find the first three chapters concerned with fundamentals, braids, bridge presentations, torus knots and pretzel knots. Chapter 3 discusses aspects of compositions and decompositions of knots and links via direct sums and tangles. Chapters 4 and 5 discuss the construction and application of spanning surfaces of knots and links. In Chapter 5 the Seifert pairing and Arf invariant are introduced and applied to the calculation of the homology groups of cyclic coverings of the knot complement. In Chapter 6 the fundamental group is introduced and used to study knots and links. The chapter ends with a brief discussion of link homotopy and the Milnor μ invariants. Chapter 7 discusses multivariable Alexander polynomials via branched covering spaces and via the Fox calculus. In Chapter 8 the Jones polynomial is introduced, first by way of the bracket polynomial state summation model. Then the notion of skein identities is introduced, and the Conway, Homflypt and Kauffman polynomials are discussed. The chapter ends with a description of a state model for a specialization of the Homflypt polynomial that is based on a solution to the Yang-Baxter equation, but this source of the model (due originally to Vaughan Jones) is not mentioned. Chapter 9 is a brief discussion of the construction of the Homflypt polynomial via representations of Hecke algebras.

Chapter 10 deals with symmetries of knots, using the knot polynomials and geometric techniques to obtain specific results. The chapter ends with a useful discussion of the decomposition work of Bonahon and Siebenmann.

Chapter 11 discusses transformations of knot diagrams that can be used to achieve transformation and unknotting.

Chapter 12 is an introduction to knot cobordism with a useful discussion of unsolved problems.

Chapters 13 and 14 begin a discussion of the embeddings of spheres in four dimensional space. These are called "2-knots" and form the first significant generalization of knot theory beyond dimension three.

Chapter 15 discusses the knot theory of embedded graphs in three dimensional space. This is a subject of interest for applications to chemistry and biology and for interconnections with graph theory. This reviewer notes one slip in the references here: the correct reference [11] to Theorem 15.2.8 (*Any embedded image of the 7-complete graph K_7 into R^3 contains a non-trivial constitutent knot*) is missing.

Chapter 16 discusses the basic formalism of the Vassiliev-Gusarov invariants and relationships with the Jones polynomial and with the Kontsevich integrals. This is the last chapter of the book before the appendices.

As the reader can see from this description of *A survey of knot theory*, this is a useful reference book that can serve as an introduction to many topics in the modern theory of knots. In terms of the developments of the past fifteen years this book does not treat the connections with statistical mechanics, quantum groups and invariants of three manifolds. On the other hand, there is a concise description of Vassiliev invariants, and this topic can lead the interested reader (through the references) to many of the topics in quantum topology. I regard this book as an indispensable addition to any library of books on knots.

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Louis H. Kauffman
University of Illinois at Chicago
E-mail address: kauffman@uic.edu