BOOK REVIEWS

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Poincaré's review of Hilbert's *Foundations of geometry*, by David Hilbert

What are the fundamental principles of geometry? what is its origin? its nature? its scope? These are questions which have at all times engaged the attention of mathematicians and thinkers, but which about a century ago took on an entirely new aspect, thanks to the ideas of Lobachevsky and of Bolyai.

For a long time we attempted to demonstrate the proposition known as the *postulate of Euclid*; we constantly failed; we know now the reason for these failures. Lobachevsky succeeded in building a logical edifice as coherent as the geometry of Euclid, but in which the famous postulate is assumed false, and in which the sum of the angles of a triangle is always less than two right angles. Riemann devised another logical system, equally free from contradiction, in which this sum is on the other hand always greater than two right angles. These two geometries, that of Lobachevsky and that of Riemann, are what are called the *non-euclidean geometries*. The postulate of Euclid then cannot be demonstrated; and this impossibility is as absolutely certain as any mathematical truth whatsoever—a fact which does not prevent the Académie des Sciences from receiving every year several new proofs, to which it naturally refuses the hospitality of the *Comptes rendus*.

Much has already been written on the non-euclidean geometries; once they scandalized us; now we have become accustomed to their paradoxes; some people have gone so far as to doubt the truth of the postulate and to ask whether real space is plane, as Euclid assumed, or whether it may not present a slight *curvature*. They even supposed that experiment could give them an answer [250] to this question. Needless to add that this was a total misconception of the nature of geometry, which is not an experimental science.

But why, among all the axioms of geometry, should this postulate be the only one which could be denied without offence to logic? Whence should it derive this privilege? There seems to be no good reason for this, and many other conceptions are possible.

However, many contemporary geometers do not appear to think so. In recognizing the claims of the two new geometries they feel doubtless that they have gone to the extreme limit of possible concessions. It is for this reason that they have conceived what they call *general geometry*, which includes as special cases the three systems of Euclid, Lobachevsky, and Riemann, and does not include any other.

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And this term *general* indicates clearly that, in their minds, no other geometry is conceivable.

They will lose this illusion if they read the work of Professor Hilbert. In it they will find the barriers behind which they have wished to confine us broken down at every point.

To understand well this new attempt we must recall what has been the evolution of mathematical thought for the last hundred years, not only in geometry, but in arithmetic and in analysis. The concept of number has been made more clear and precise; at the same time it has been generalized in various directions. The most valuable of these generalizations for the analyst is the introduction of *imaginaries* which the modern mathematician could not now dispense with; but we have not stopped with this; other generalizations of number, or, as we say, other categories of complex numbers, have been introduced into science.

The operations of arithmetic have in their turn been subjected to criticism, and Hamilton's quaternions have given us an example of an operation which presents an almost perfect analogy to multiplication, and may be called by the same name, which, however, is not commutative, that is, the product of two factors is not the same when the order of the factors is reversed. This was a revolution in arithmetic quite comparable to that which Lobachevsky effected in geometry.

Our conception of the infinite has been likewise modified [251]. Professor G. Cantor has taught us to distinguish gradations in infinity itself (which have, however, nothing to do with the infinitesimals of different orders invented by Leibniz for the ordinary infinitesimal calculus). The concept of the continuum, long regarded as a primitive concept, has been analyzed and reduced to its elements.

Shall I mention also the work of the Italians, who have endeavored to construct a universal logical symbolism and to reduce mathematical reasoning to purely mechanical rules?

We must recall all this if we wish to understand how it is possible that conceptions which would have staggered Lobachevsky himself, revolutionary as he was, can seem to us to-day almost natural, and can be propounded by Professor Hilbert with perfect equanimity.

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