# BOOK REVIEWS

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*Selected works of Ellis Kolchin*, by H. Bass, A. Buium, and P.J. Cassidy (editors), American Mathematical Society, Providence, RI, 1999, xiii + 639 pp., $120.00, ISBN 0-8218-0542-8

"Collected Works" is a standard genre for mathematics books, along with monographs, textbooks, lecture notes, and conference proceedings. But unlike these others, the value of ownership to an individual mathematician is not clear. Among the several hundred mathematics books in my office are exactly three collected works. One, and the only one I've owned for very long, is the *Collected Works of Abel*, edited by Sylow and Lie and printed in 1881 by the Norwegian Mathematical Society. I purchased this, from the publisher, as a souvenir during a 1970 visit to Oslo for an algebraic geometry conference. That it was still for sale 90 years after it was printed, in the original edition at what must have been the original price (it was very cheap), is some evidence about the commercial potential of collected works. Another of the collected works volumes I own I found on the sale table in a discount bookstore in a rural outlet mall, selling for less than 5% of its commercial publisher's list price. Since in addition I know and respect the author, that was an easy purchase decision. And as for the third: I needed to use up some book purchase credits, and a title in the AMS Collected/Selected Works series fit. While all these acquisitions are mathematically grounded, and satisfying from various points of view, including pride of possession, I can't say that I've read them in any professional mathematical way.

I doubt that many collected works are so read. But the *Selected Works of Ellis Kolchin with Commentary* may be an exception. To paraphrase slightly the first sentence of the editors' preface, differential algebra, especially differential Galois theory in its modern rigorous algebraic incarnation, is mostly the creation of Ellis Kolchin. In addition to his published papers, Kolchin presented his results in definitive form in two major research monographs [6], [7] (not reprinted in this volume). These lay out their themes – differential Galois theory and differential algebraic groups – in masterful generality and establish all the important results in the widest possible context. But as a consequence, for all their definitiveness,

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the research monographs are not a common choice for self-study, especially for beginners.[1]

But a beginner could be well served by going directly to Kolchin's original papers reprinted here: [8], which covers the Picard – Vessiot differential Galois theory, or the Galois theory of differential fields in which the Galois group is a linear algebraic group; and [9], which covers the strongly normal differential Galois theory, or the Galois theory of differential fields in which the differential Galois group is a linear algebraic group.

A simple example of a Picard – Vessiot extension is given by starting with the field $F = \boldsymbol{C}(t)$ of rational functions of one complex variable with derivation $D_F = \frac{d}{dt}$ and then forming an extension by adjoining an antiderivative, say $y = \log(t) = \int \frac{1}{t}$ and/or an exponential $w = e^t$ to form the differential field extension $E = F(y, w) \supset F$. (The derivation $D_E$ is given by $D_E(y) = \frac{1}{t}$, $D_E(w) = w$, and $D_E(f) = D_F(f)$ for $f \in F$.) Differential automorphisms of $F(w)$ over $F$ must carry $w$ to a non-zero complex scalar multiple $cw$ of itself, $c \in \boldsymbol{C}$, and this determines the automorphism. Moreover, every $c$ occurs. So there is a bijection between the group $G(F(w)/F)$ of differential automorphisms of $F(w)$ over $F$ and the multiplicative group $\boldsymbol{G}_m(\boldsymbol{C})$. Similarly, since differential automorphisms of $F(y)$ over $F$ must carry $y$ to $y+d$ for some complex scalar $d$, it turns out that there is a bijection between $G(F(y)/F)$ and the additive group $\boldsymbol{G}_a(\boldsymbol{C})$. What makes $F(w) \supset F$ (or $F(y) \supset F$) a Picard – Vessiot extension is that the top field is generated, as a differential field, over the bottom field by a full set of solutions of a linear homogenous differential equation over the bottom field ($W' - W = 0$ and $Y'' + tY' = 0$ respectively); "full" means the dimension over the constants of the set of solutions in the extension is the order of the equations. "Constants" are the elements of derivative zero, and it is also required that every constant of the extension field already lie in the base. It is generally assumed that the constants form a subfield which is algebraically closed and of characteristic zero.

A tower of Picard – Vessiot extensions need not be Picard – Vessiot (for example $\boldsymbol{C}(t, \log(t), \log(\log(t))) \supset \boldsymbol{C}(t, \log(t)) \supset C(t)$). A tower of Picard – Vessiot extensions in which the differential Galois group of each successive layer is either a $\boldsymbol{G}_m$ or a $\boldsymbol{G}_a$ [2] is called a *Liouville* extension. One of the main application results in [8] is that a Picard – Vessiot extension embeds in a Liouville extension if and only if the differential Galois group of the Picard – Vessiot extension is solvable. Since having group $\boldsymbol{G}_a$ comes from adjoining an antiderivative, and having group $\boldsymbol{G}_m$ comes from adjoining an exponential, another way to state this theorem is to say that a full set of solutions to a differential equation can be obtained from repeated adjunction of exponentials and antiderivatives if and only if the associated differential Galois group is connected solvable.

An example of a strongly normal extension which is not Picard – Vessiot is given by starting with $F = \boldsymbol{C}(t)$ again and adjoining a Weierstrauss $p$ function $y$ and its derivative: $E = F(y, y') \supset F$. Here the differential Galois group $G(E/F)$ of $E$ over $F$ turns out to be the one-dimensional abelian variety (elliptic curve) associated with $y$.

[1] The books also use the language of Weil's *Foundations* [17], which is less familiar to students taught scheme-based algebraic geometry.

[2] or finite.

We stated above that a beginner could be well served by plunging directly into [8] and [9]. But this volume offers more: it includes a paper by A. Borel [2], one of four terrific contemporary expository and survey articles included by the editors in the final section of this volume, entitled "Commentary". Borel explains Kolchin's interest in and work on the theory of algebraic groups over arbitrary fields. He also gives an excellent summary of Kolchin's Galois theory, both the Picard – Vessiot case and the far more complicated general strongly normal case. (The complications come about from the need to consider isomorphisms of the extension $E \supset F$ into some universal differential field $U \supset F$ containing $E$; "universal" means that all finitely generated differential field extensions of differential subfields of $U$ finitely generated over $F$ embed in $U$.) A beginner who starts with [2] and then goes on to [8] and [9] will be a beginner no longer.

Kolchin was also somewhat interested in special cases and applications of differential Galois theory, and his publications on these matters are included in the volume. However, readers interested in these topics will get a much more complete, and up to date, overview from M. Singer's paper [15] in the "Commentary" section. Singer surveys the direct and inverse problems of [Picard – Vessiot] differential Galois theory: "direct" means "given a linear homogenenous differential equation, determine the corresponding linear algebraic differential algebraic group"; "inverse" means "given a linear algebraic group, find a differential equation with that group as differential Galois group".[3] (In both cases a base differential field is specified.) And for both problems, constructive methods are sought.

There is no complete solution to the direct problem, in the sense of an algorithm which calculates the differential Galois group $G(E/F)$ for a Picard – Vessiot extension $E$ for the differential equation $L(Y) = 0$ over the field $F$, even when $F$ is a constructible field like $\overline{\boldsymbol{Q}}(t)$. It is not even known how to determine if $G(E/F)$ is trivial or finite. However, there are many solutions for parts of the direct problem when further assumptions are made about either $G$ (such as that it is virtually solvable) or $L$ (such as that it has order two) and, in many cases, when assumptions can be made about the action of $G(E/F)$ on the set of solutions $V = \{y \in E \mid L(y) = 0\}$ of $L$ in $E$. (We recall that one of the conditions for Picard – Vessiot was that $V$ be full, namely a vector space over the field of constants $C$ of $F$ and $E$ of dimension the order of $L$.) $G = G(E/F)$ is an algebraic group over $C$ and its action on $V$ is algebraic (and faithful). When the action of $G$ on $V$, for example, is reducible, and especially in the case when $V$ has one dimensional submodules or even summands, and the dimension of $G$ is small, then algorithims are known. More generally, from the pair $G, V$ one can generate the tensored abelian subcategory of the category of all algebraic $G$ modules. By the theory of Tannakian categories, this category determines $G$, and thus assumptions about the action of $G$ on $V$ can sometimes determine $G$.

Tannakian categories were introduced into differential Galois theory in 1980 by P. Deligne [4]; see below.

The inverse problem was traditionally studied over the base field $F = \boldsymbol{C}(t)$. As C. Tretkoff and M. Tretkoff pointed out [16], the fact that every linear algebraic $G$ group over $\boldsymbol{C}$ is a differential Galois group over $F$ is a consequence of analytic

[3]As Singer points out, in his 1966 lecture at the International Congress in Moscow, Kolchin identified these as the main outstanding problems in Picard – Vessiot theory. The lecture is reprinted in this volume.

existence theorems. Algebraic proofs for special sorts of connected groups have been given, and in [13] Singer and C. Mitschi give an algebraic proof for all connected groups. The situation for arbitrary groups, and in particular for finite groups, is still open: here the problem is the same as exhibiting them as ordinary Galois groups over $F$, which is known analytically (Riemann existence theorem) but not in general algebraically.

For some other base fields, the inverse problem can be solved as well. J. Ramis [12], for instance, gave a solution for convergent power series over $\boldsymbol{C}$.

As noted above, once Kolchin began to consider Galois theory for differential field extensions $E \supset F$ with $G(E/F)$ not necessarily linear, he had to employ a universal differential extension $U \supset F$. This contains isomorphic copies of all finitely generated differential extensions of finitely generated differential extensions of $F$ in $U$. This naturally leads to concepts of "differentially closed" field and of "differential closure".

These latter, as it happens, have been fruitful notions in logic as well as differential algebra. The paper by B. Poizat [14] in the "Commentary" section explains this well. The sections on universal differential fields, differentially closed fields, and differential closure in the "Commentary" paper by A. Buium and P. Cassidy [3] are also a good introduction. Or the reader could begin first with Kolchin's paper [10], whose introduction contains the goal of a "unified exposition of the whole theme, in the setting and language of differential algebra, with the aim of making as explicit as possible the several ideas involved and the relations among them."

The paper by Buium and Cassidy also does much more: it provides an introduction to the final major theme in Kolchin's work, namely a theory of differential algebraic groups and differential algebraic geometry. This volume includes the brief notes of Kolchin's 1975 AMS Colloquium lectures on differential algebraic groups, his 1978 Soviet Academy of Science paper on differential algebraic groups, and the notes of his 1984 Beijing lectures on the same subject. His definitive work here, however, was published in his book *Differential Algebraic Groups*[4] [7]. Buium and Cassidy's essay gives an excellent introduction to this subject, as well as the broader subject of differential algebraic geometry, in terms that the reader unfamiliar with the details of Kolchin's theory can understand. Among other things, this enables the reader to appreciate why Kolchin thought this aspect of the subject would be important, and to appreciate the many significant acomplishments (and the applications) of the work of Buium and of Cassidy as well.

Differential algebra, including differential Galois theory, has grown and broadened in recent years. Even to recount some current developments just in the topics that were Kolchin's particular interests is far beyond the scope of this review. Fortunately, the "Commentary" papers of Singer, Poizat, and Buium and Cassidy, with their collective bibliographies of nearly 340 entries, do this well. And of course there is much other work not referenced as well. Nonetheless, we would like to close with a couple of observations on approaches to differential Galois theory.

There seems to be general agreement among workers in the subject that Deligne's version, via the Tannakian category theory, is the most promising way to view the Picard – Vessiot theory. The idea here is to construct a Picard – Vessiot extension of the field $F$ for the differential operator $L$ over $F$ by first associating to $F, L$ a

[4]This reviewer, who reviewed the book for both *Mathematical Reviews* and *Zentralblatt für Mathematik*, stated, "[t]his is a demanding book." It still is.

finite dimensional $F$-vector space $W$ with a connection $\nabla$, such that the kernel of $\nabla$ has a full set of solutions of $L = 0$ for a basis. Then one takes the tensored subcategory of the tensored catgory of finite dimensional $F$-vector spaces with connections generated by $(W, \nabla)$. This subcategory is Tannakian, and so there is a linear algebraic group canonically associated with it, which turns out to be the differential Galois group, and the Picard – Vessiot extension is the function field of (a twisted form of) this group. One can even drop the operator $L$ and just begin with any $F$-vector space with a connection. Singer's paper in this volume includes a sketch of this theory; there is also a sketch in D. Bertrand's review [1], and, for those comfortable with the language of schemes, there is Deligne's original paper [4]. Most detailed expositions of the Picard – Vessiot theory aimed at students, however, such as [11], use pre–Tannakian methods.

The algebraic groups associated to Tannakian categories are linear. Thus this approach must be limited to the Picard – Vessiot theory.

Less exploited to date has been the connection between the general Galois theory in categories and differential Galois theory. Some years ago, G. Janelidzhe [5] showed that Picard – Vessiot differential Galois theory was a special case of Galois theory in categories and used this to give a construction of the differential Galois group in the Picard - Vessiot case. It is possible that this approach as well might serve to capture the strongly normal theory.

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