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The q-Schur algebra, by Stephen Donkin, London Mathematical Society Lecture Note Series, vol. 253, Cambridge Univ. Press, Cambridge, 1999, x + 179 pp., $39.95, ISBN 0-521-64558-1

Iwahori-Hecke algebras and Schur algebras of the symmetric group, by Andrew Mathas, University Lecture Series, vol. 15, American Mathematical Society, Providence, RI, 1999, xiii + 188 pp., $25.00, ISBN 0-8218-1926-7

Schur algebras consist of certain families of finite dimensional algebras which connect the representation theories of the symmetric groups $\mathfrak{S}_r$ and the general linear groups $GL(n,k)$ over a field k. When $k = \mathbb{C}$, the complex numbers, the theory goes back to the early years of the last century (i. e., the early development of representation theory). For example, using the determination of the complex irreducible characters of the symmetric groups by G. Frobenius, I. Schur worked out the polynomial representation theory for the groups $GL(n,\mathbb{C})$, establishing complete reducibility and calculating the irreducible characters. H. Weyl further highlighted this point of view in his famous book [W], extending it to other classical groups. Subsequently, this whole approach often came to be called "Schur-Weyl-Frobenius reciprocity". (For references and further discussions, see C. Curtis' book [C] on the history of representation theory.)

In modern times, J. A. Green's Yale lectures [G], published in 1980, presented a theory relating Schur algebras over fields k of arbitrary characteristic and the modular representation theory of symmetric groups. His notes still remain a good introduction to this branch of representation theory. Over the past several decades, this theory of Schur algebras expanded in an important direction, coincidentally with the rise of "q-mathematics". For the purposes of this review, we can informally think of the latter subject as involving situations in which a finite group algebra, algebra of functions on a Lie group, etc., is deformed by introducing a parameter q. Historically, q-Schur algebras first arose in the work of M. Jimbo [J] in connection with physics. Independently, R. Dipper and G. James [DJ] introduced q-Schur algebras over general fields k in connection with the modular "non-describing characteristic" representation theory of the finite general linear groups $GL(n,\mathbb{F})$. To be more precise, this involves the representation theory of $GL(n,\mathbb{F})$ over fields k of characteristic p not dividing the order of the finite field $\mathbb{F}$. This connection with the modular representation theory of finite general linear groups probably remains today the most important application of the theory of q-Schur algebras over general fields.

We will give several descriptions of q-Schur algebras, beginning with the one related to Hecke algebras of symmetric groups. First, we introduce some definitions. Fix a positive integer r, and let S be the set of fundamental "reflections" $(j, j+1)$, $j < r$, in the symmetric group $\mathfrak{S}_r$. For $s, s' \in S$, let $m(s, s')$ be the order of ss'. The (Iwahori-)Hecke algebra $\mathcal{H} = \mathcal{H}(\mathfrak{S}_r, k, q)$ associated to $\mathfrak{S}_r$ is the finite dimensional

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k-algebra defined by generators T_s, $s \in S$, and relations

$$(1)\qquad \begin{cases} T_s^2 = q \cdot 1 + (q-1)T_s \\ \underbrace{T_s T_{s'} T_s \cdots}_{m(s,s')} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m(s,s')} \end{cases}$$

over all distinct $s, s' \in S$. Here q is an arbitrary non-zero element in the field k. For example, if we take $q = 1_k$, the identity element of k, then $\mathcal{H}$ is just the group algebra $k\mathfrak{S}_r$. In connection with the finite group $GL(n, \mathbb{F})$, we would take $q = |\mathbb{F}| \cdot 1_k$; in this case, $\mathcal{H}$ is isomorphic to the endomorphism algebra of the permutation module for $GL(n, \mathbb{F})$ defined by a Borel subgroup. The algebra $\mathcal{H}$ is an example of the kind of deformation of $k\mathfrak{S}_r$ mentioned above. While the two books under review focus primarily on the theory of Hecke algebras associated to symmetric groups, these algebras can be defined for any Coxeter group W. When $k = \mathbb{C}$ there exists a vast literature relating the theory of Hecke algebras to the representation theories of complex Lie algebras, finite groups of Lie type, ... See [KL], [KT], [CR], ... In addition, the methods here often involve sophisticated geometric theories (e. g., intersection cohomology).

Now let V be a vector space of dimension n over k with fixed ordered basis $v_1, \dots, v_n$. For a sequence $I = (i_1, \dots, i_r)$ of integers i_j, $1 \le i_j \le n$, put $v_I = v_{i_1} \otimes \cdots \otimes v_{i_r}$. For $\sigma \in \mathfrak{S}_r$ write $I\sigma = (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(r)})$. The tensor space $V^{\otimes r}$ carries the structure of a right $\mathcal{H}$-module by setting, for $s = (j, j+1) \in S$,

$$(2)\qquad v_I T_s = \begin{cases} q v_{Is}, & \text{if } i_j \le i_{j+1}; \\ v_{Is} + (q-1) v_I, & \text{otherwise,} \end{cases}$$

since it can be verified that the relations (1) hold for the operators on $V^{\otimes r}$ defined in (2). Taking $q = 1_k$, $\mathcal{H} \cong k\mathfrak{S}_r$, the group algebra over k of $\mathfrak{S}_r$, and (2) is the usual permutation action. In general, the q-Schur algebra $S_q(n, r)$ can then be defined as the endomorphism algebra

$$(3)\qquad S_q(n, r) = \mathrm{End}_{\mathcal{H}}(V^{\otimes r}).$$

Thus, if $q = 1_k$, then $S_1(n, r) \cong S(n, r)$, the classical Schur algebra over k. Much of this formalism works in a characteristic-free setting, replacing $\mathcal{H}$, etc. by the corresponding objects over the ring $\mathbb{Z}[q, q^{-1}]$ of integer Laurent polynomials in a variable q. Because $V^{\otimes r}$ is a direct sum of "q-permutation modules", it behaves nicely with respect to base change.

The tensor space $V^{\otimes r}$ provides a natural connection between the module categories for $S_q(n, r)$ and $\mathcal{H}$. For example, when $n \ge r$, $V^{\otimes r} \cong S_q(n, r)e$ for an idempotent $e \in S_q(n, r)$. In this case, $\mathcal{H} \cong eS_q(n, r)e$, and the *Schur functor*

$$(4)\qquad F : S_q(n, r) - \mathrm{mod} \longrightarrow \mathcal{H} - \mathrm{mod}, \quad M \mapsto eM,$$

passes from $S_q(n, r)$-modules to $\mathcal{H}$-modules. In the most classical case, when $q = 1$ and $k = \mathbb{C}$, Schur used (4) to relate the representation theory of $\mathfrak{S}_r$ with that of $GL(n, \mathbb{C})$ using the Schur algebras $S(n, r)$ as intermediaries. For general q, we need to introduce a new player—the quantum general linear groups—as a replacement for $GL(n, \mathbb{C})$. This object can be understood first in terms of quantum matrix space $M_q(n)$. The most familiar version [M] has as coordinate algebra $k[M_q(n)]$

the algebra generated by n^2-variables X_{ij} satisfying the relations[1]

$$(5)\qquad \begin{cases} X_{ri}X_{rj} = q^{-1}X_{rj}X_{ri}, & i < j; \\ X_{ri}X_{si} = q^{-1}X_{si}X_{ri}, & r < s; \\ X_{ri}X_{sj} = X_{sj}X_{ri}, & r < s, i > j; \\ X_{ri}X_{sj} - X_{sj}X_{ri} = (q^{-1} - q)X_{si}X_{rj}, & r < s, i < j. \end{cases}$$

Then $k[M_q(n)]$ is a bialgebra with comultiplication defined by

$$(6)\qquad \Delta(X_{ij}) = \sum_l X_{il} \otimes X_{lj}.$$

Further, the *quantum determinant*

$$(7)\qquad \det_q = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-\ell(\sigma)} X_{1\sigma(1)} \cdots X_{n\sigma(n)}$$

is a central group-like element in $k[M_q(n)]$. The bialgebra $k[M_q(n)][1/\det_q]$ then becomes a Hopf algebra (with antipode defined by Cramer's rule) which serves as the coordinate algebra $k[GL_q(n)]$ of the (Manin) quantum group $GL_q(n) = GL_q(n,k)$. Again, when $q = 1$, $GL_q(n)$ identifies with the usual general linear group $GL(n,k)$, and $k[GL_q(n)]$ represents a deformed version of the usual coordinate algebra of functions on $GL(n,k)$.

As with quantum matrix space $M_q(n)$, the idea is that $GL_q(n)$ and its representation theory are completely determined by its coordinate algebra $k[GL_q(n)]$; one should not think of $GL_q(n)$ as an actual group (which it is not). For example, by analogy with affine algebraic groups, a rational $GL_q(n)$-module consists of a comodule for the Hopf algebra $k[GL_q(n)]$; i. e., it is a vector space V together with a linear map $V \to k[GL_q(n)] \otimes V$ satisfying certain natural properties. More for our purposes, the representation theory of $GL_q(n)$ can be conveniently reduced to the study of the representation theory for the q-Schur algebras $S_q(n,r)$, $r \geq 1$, over k. To see why leads to another description of the q-Schur algebras.

Let $A_q(n,r)$ be the subspace of $k[M_q(n)]$ spanned by monomials in the X_{ij} (relative to some fixed ordering of these variables) of degree r. The form of the comultiplication Δ on $k[M_q(n)]$ shows that $A_q(n,r)$ is a subcoalgebra of $k[M_q(n)]$. It has finite dimension equal to $\binom{n^2+r-1}{r}$. Thus, the dual Δ^* defines an algebra structure on the dual space $A_q(n,r)^*$. In fact,

$$(8)\qquad S_q(n,r) \cong A_q(n,r)^*.$$

Any $S_q(n,r)$-module M has a natural $A_q(n,r)$-comodule structure and hence is a comodule for $k[GL_q(n)]$. Thus, M is a rational $GL_q(n)$-module—a so-called *polynomial representation.* Conversely, any indecomposable rational $GL_q(n)$-module can, after twisting by some power of $\det_q$, be viewed as a polynomial representation.

When $\mathbb{F}$ is a finite field and $q = |\mathbb{F}| \cdot 1_k$, the q-Schur algebra $S_q(n,n)$ over k has yet a third interpretation in the spirit of (3), but this time involving an endomorphism

[1] Although we cannot go into it here, these strange-looking relations do have explanations, e. g., one coming out a solution to the famous quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} : V^{\otimes 3} \to V^{\otimes 3}$; see [CP].

algebra for the finite general linear group $GL(n,\mathbb{F})$. Namely, we have

$$S_q(n,n) \cong \mathrm{End}_{kGL_n(\mathbb{F})}\big(\bigoplus_{P\supseteq B} kGL(n,\mathbb{F})/P\big). \tag{9}$$

The direct sum in (9) is over all parabolic subgroups P containing a fixed Borel subgroup B of $GL_n(\mathbb{F})$, while $kGL(n,\mathbb{F})/P$ is just the permutation module on the cosets of P. Putting all the pieces together eventually leads to a connection between the non-describing representation theory of the finite general linear groups $GL(n,\mathbb{F})$ over a field k and the representation theory of the quantum groups $GL_q(n,k)$ for $q = |\mathbb{F}|\cdot 1_k$. Questions about decomposition numbers, cohomology, etc. for $GL(n,\mathbb{F})$ can be reformulated in terms of similar questions for $GL_q(n,k)$ and then sometimes recast again "generically", i. e., for large values of the parameters, into characteristic zero problems; see, e. g., [DJ], [CPS]. For the other finite groups G of Lie type, analogs of the q-Schur algebra have been investigated, but the connection with quantum groups remains generally open.

As their titles suggest, both books under review concern the topics introduced above, yet each takes a markedly different approach to the subject. Donkin's tack is "to first prove results about our quantum version of GL_m, then to use this knowledge to deduce results about the q-Schur algebras and finally, by a further 'descent' to obtain results on the Hecke algebra" (p. x). Mathas, on the other hand, states that his "notes adopt the view that the Iwahori-Hecke algebras—rather than the q-Schur algebras—are the objects of central importance. This is partly a matter of personal taste and partly expedience; other authors ... travel in the reverse direction" (p. x).

The reader not already somewhat familiar with the theory of q-Schur algebras should be forewarned that Donkin's book is not an introduction to that theory. In fact, the author views his book largely as a continuation of his research article [D], which takes a decidedly homological view of the subject (as opposed to, say, the combinatorial approach for Schur algebras laid down by Green [G]). But to make the material accessible to a wider audience, Donkin has included a preliminary chapter which collects together the necessary background material. The exposition moves swiftly here. For example, the q-analog of Kempf's vanishing theorem receives a one-sentence treatment. The q-Schur algebras are defined from the quantum group along the lines of (8). Later chapters treat a number of specific advanced topics, e. g., the bideterminant basis of $A_q(n,r)$, the classification of the irreducible $\mathcal{H}$-modules using the Schur functor (4), the analog of the Steinberg tensor product theorem, the theory of tilting modules for q-Schur algebras, the determination of the precise global homological dimension of q-Schur algebras, etc. The book contains an interesting section on the 0-Schur algebra, in which Donkin completely determines its irreducible characters. A useful tool in some of this is the quasi-heredity of the q-Schur algebras, and the book concludes with a brief appendix outlining the elementary features of quasi-hereditary algebras. From different points of view, most of the results in these notes already exist in the literature, sometimes in more general (e. g., characteristic-free) form and with shorter proofs, a fact the author does not always document carefully. It is, however, nice to have things collected together and carefully proved from the author's particular point of view.

Despite the above strengths, Donkin's book presents some obstacles to the reader. First, it contains little motivation for the study of q-Schur algebras, apart from their application to Hecke algebras (which themselves are left unmotivated). Perhaps this fact is in keeping with its "terse journal style" (p. x), but, in this sense, the book does not compare favorably to Mathas' book discussed below. Also, some readers will be familiar with the quantum group defined by (8), while this book uses another version introduced in [DD]. A brief mention, without details, of [AST] or [DPW] would have been useful since these papers show how the various quantum deformations of the general linear group lead to the same q-Schur algebras. Without this fact, the reader may rightly wonder how the theory in this book applies to q-Schur algebras in the literature. Nor does the author hint at any connections of q-Schur algebras with quantum enveloping algebras. Finally, the historical notes on quasi-hereditary algebras skew that development by omitting mention of the considerable influence of the geometric theory of perverse sheaves in their invention and development.

In the second book under review, Mathas surveys a wide range of topics related to the Hecke algebras $\mathcal{H} = \mathcal{H}(\mathfrak{S}_r, k, q)$. The author develops and makes systematic use of the theory of cellular algebras as defined in [GL]. Such algebras have a particularly nice basis—in fact, the theory might be thought of as a natural abstraction of properties of the Kazhdan-Lusztig basis for $\mathcal{H}$ [KL] and the Robinson-Schensted correspondence between elements of $\mathfrak{S}_r$ and pairs (S, T) of standard tableaux. The present book, however, shows that $\mathcal{H}$ is cellular by means of the so-called Murphy basis for $\mathcal{H}$. This leads to a classification of the irreducible $\mathcal{H}$-modules, and, defining q-Schur algebras $S_q(n, r)$ in the spirit of (3) above, a classification of the irreducible modules for these algebras as well. (The approach contrasts with Donkin's, who first classifies the irreducible modules of $S_q(n, r)$ using highest weight theory, then gets the classification of the irreducible $\mathcal{H}$-modules via the Schur functor.) Using the cellular property of the Murphy basis, Mathas also proves that $S_q(n, r)$ is quasi-hereditary. Next, utilizing "algebraic group methods", the author determines the blocks for $S_q(n, r)$, $n \geq r$, and, by means of (4), obtains the blocks for $\mathcal{H}$, the q-analog of the Nakayama conjecture for $\mathfrak{S}_r$. The final, longest, and most interesting chapter surveys, usually without proofs, a number of more recent results. For example, the chapter discusses in some detail the LLT algorithm (after Lascoux, Leclerc and Thibon [LLT] and its proof by Ariki [A]) for calculating the decomposition matrix of the complex Hecke algebra $\mathcal{H}(\mathfrak{S}_r, \mathbb{C}, q)$ for q a root of unity. Other topics include: the modular branching work of Kleshchev and Brundan for determining the socle of the restrictions of irreducible $\mathcal{H}$-modules to Hecke subalgebras, a sketch of the theory of Ariki-Koike algebras in which $\mathfrak{S}_r$ is replaced by a certain complex reflection group, and a short (and not too enlightening) introduction to the non-describing representation theory. The book concludes with some useful tables.

Mathas' book contains many exercises which introduce the reader to a number of further interesting topics (e. g., the Robinson-Schnested correspondence). Thus, students will find it useful. Historical notes at the end of each chapter provide some context for the discussion. Written with considerable enthusiasm, the book contains occasional slips. For example, the assertion on p. 9 that the function field $\mathbb{C}(t)$ is algebraically closed does not inspire much confidence. Nor does the claim on p. 17 (repeated in Exercise 2.7) that the left cell module $C^{*\lambda}$ is isomorphic to the linear dual $\operatorname{Hom}_R(C^\lambda, R)$ of the right cell module C^λ in a cellular algebra A over a commutative ring R. As these concepts are defined here, this claim already fails for the Specht module $C^{(2,1)}$ of symmetric group $\mathfrak{S}_3$ when R is a field of characteristic 3. Also, parts of the book unfortunately have the feel of something quickly put together. For example, in the context of discussing the Grothendieck group of Hecke algebras, what should one make of muddled statements such as the definition of the complexified Grothendieck group "as the additive abelian group (with complex coefficients) generated by the symbols . . . " on p. 97, and the resulting confusion in Corollaries (6.4) and (6.5) which, as stated, have little substance? (Perhaps these corollaries should be definitions?)

Taken together these two books cover some of the same topics, but from different perspectives. Both books adopt a strictly algebraic/combinatorial point of view, although Mathas at least mentions that geometry does come into the picture. The beginner will find the Mathas book helpful, while those who already know something about q-Schur algebras will appreciate Donkin's approach.

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