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Mirror symmetry and algebraic geometry, by David A. Cox and Sheldon Katz, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, xxi + 469 pp., \$69.00, ISBN 0-8218-1059-6

In 1991 the physicists Candelas, de la Ossa, Green and Parkes published a famous paper [3] which contained some astonishing predictions about rational curves on the quintic threefold in $\mathbb{P}^4$. These predictions were obtained using a mysterious "mirror symmetry" for Calabi-Yau threefolds, and they went far beyond anything algebraic geometry could prove at the time. For this reason the paper [3] became a challenge for mathematicians to understand mirror symmetry and to find a mathematically rigorous proof of the predictions made by physicists. The process of creating a rigorous mathematical foundation for mirror symmetry is still far from being finished. However, after the works of Givental [6], [7] and the paper of Lian, Liu and Yau [10], it became clear that the first period of this process is already over. During this period mirror symmetry has given impetus to new fields of algebraic geometry. The primary goal of the book is to give an introduction to these algebro-geometric aspects of mirror symmetry.

1. Mirrors of quintic threefolds

Let us consider the power series

$$y_0(x) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}(-1)^n x^n = 1 - 120x + \cdots .$$

If we put $a_n = (-1)^n (5n)!/(n!)^5$, then the numbers a_n satisfy the recurrent relation

$$(n+1)^4 a_{n+1} = -5(5n+1)(5n+2)(5n+3)(5n+4)a_n.$$

It follows immediately from this relation that $y_0(x)$ is a solution to the differential equation $\mathcal{D}y = 0$, where

$$\mathcal{D} = \Theta^4 + 5x(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4), \quad \Theta = x\frac{d}{dx}.$$

The equation $\mathcal{D}y = 0$ has a regular singular point at $x = 0$. There is another solution

$$y_1(x) = y_0(x)\log(-x) + 5\sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\left[\sum_{j=n+1}^{5n} \frac{1}{j}\right](-1)^n x^n$$

with a logarithmic singularity at $x = 0$. Using y_0 and y_1, one can define near $x = 0$ another local coordinate $q := \exp(y_1/y_0)$. The mirror symmetry for a "generic" quintic threefold $V \subset \mathbb{P}^4$ predicts that the function

$$K(q) := \frac{5}{(1+5^5x)y_0^2(x)}\left(\frac{q}{x}\frac{dx}{dq}\right)^3,$$

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as a function of the local coordinate q, has the following expansion:

$$K(q) = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d},$$

where n_d denotes the *instanton number* of rational curves of degree d on V. Direct computations show that

$$K(q) = 5 + 2875\frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + 317206375 \cdot 3^3 \frac{q^3}{1-q^3} + \cdots .$$

The number $n_1 = 2875$ is classically known to be the number of lines on a generic quintic threefold. It has been proved that $n_2 = 609250$ and $n_3 = 317206375$ are analogous number for conics [8] and twisted rational cubics [4]. However, in general one cannot expect that n_d gives the number of rational curves on a generic quintic threefold for all $d \geq 1$ even if one assumes that this number is finite. This shows the necessity of a rigorous mathematical definition of n_d using so-called Gromov-Witten invariants and the intersection theory on the Kontsevich moduli space of stable maps [9]. One can easily show that all n_d are rational, but so far it is still unknown if they are integers (though this is true for all n_d that have been computed).

Mirror symmetry allows us to interpret the above differential equation $\mathcal{D}y = 0$ as the Picard-Fuchs differential equation for periods of a 1-parameter family of Calabi-Yau threefolds V_x° which are called mirrors of quintic threefolds $V \subset \mathbb{P}^4$. The family of mirrors V_x° can be constructed explicitly as follows. Let G be the abelian group of order 125

$$G := \{(a_1, \dots, a_5) \in (\mathbb{Z}/5\mathbb{Z})^5 \ : \ \sum_i a_i = 0 \bmod 5\}/H,$$

where $H \cong \mathbb{Z}/5\mathbb{Z}$ is embedded diagonally. Then G acts on $\mathbb{P}^4$ as $g(x_1, \dots, x_5) = (\mu^{a_1}x_1, \dots, \mu^{a_5}x_5)$. The family V_x° consists of minimal desingularizations of hypersurfaces in $\mathbb{P}^4/G$ defined by the equation

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0, \ \ \psi \in \mathbb{C},$$

where $x = \psi^{-5}$ is the parameter of the family .

For the Hodge numbers of V and $V^\circ = V_x^\circ$ one has the equalities $h^{1,1}(V) = h^{2,1}(V^\circ) = 1$ and $h^{2,1}(V) = h^{1,1}(V^\circ) = 101$ which reflect the mirror isomorphism of the superconformal field theories associated with (V, ω) and (V°, ω°), where ω (resp. ω°) denotes a complexified Kähler class on V (resp. on V°). This isomorphism suggests that the Gauss-Manin connection of the variation of Hodge structure in $H^3(V^\circ, \mathbb{C})$ should be identified with the so-called A-connection of the A-variation of Hodge structure in $\bigoplus_{i=0}^3 H^{2i}(V, \mathbb{C})$. The precise meaning of this identification is the content of the *Mirror Theorem* proved by Givental [6] and by Lian-Liu-Yau [10]. The Mirror Theorem plays one of the central roles in the whole book. The main ingredients of its proof are explained in Chapters 9-11. The Mirror Theorem would be impossible to prove without the creation of new mathematical theories of Gromov-Witten invariants and quantum cohomology. Different mathematical approaches to these theories are discussed in detail in Chapters 7 and 8.

2. Generalizations and toric geometry

It would be more difficult to investigate mirror symmetry if quintic threefolds were the only examples of Calabi-Yau threefolds for which one could make explicit calculations of instanton numbers of rational curves and compare them with classical results from enumerative geometry. Fortunately, there exists a large class of examples for which all the above calculations can be generalized. These examples are Calabi-Yau hypersurfaces and complete intersections in toric varieties [2], [11], [12]. Moreover, toric geometry helps to express mirror symmetry in an elementary way in terms of polar duality between special convex polyhedra [1].

Let M be a free abelian group of rank d, $N := \mathrm{Hom}(M, \mathbb{Z})$ the dual group, and $\langle *, * \rangle\ M \times N \to \mathbb{Z}$ the natural pairing. A convex d-dimensional polytope $\Delta \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ is called *reflexive* if the following conditions are satisfied:

1) the origin $0 \in M$ is contained in the interior of Δ;
2) all vertices of Δ belong to the lattice $M \subset M_{\mathbb{R}}$;
3) all vertices of the *polar polytope*

$$\Delta^\circ = \{v \in N_{\mathbb{R}} := N \otimes \mathbb{R}\ :\ \langle u, v \rangle \geq -1 \text{ for all } m \in \Delta\}$$

belong to the dual lattice $N \subset N_{\mathbb{R}}$.

It is easy to see that if $\Delta \subset M_{\mathbb{R}}$ is reflexive, then $\Delta^\circ \subset N_{\mathbb{R}}$ is again reflexive and $(\Delta^\circ)^\circ = \Delta$. Therefore one obtains a natural involution on the set of all d-dimensional reflexive polyhedra. It turns out that this involution has a direct relation to mirror symmetry. Let us consider elements $m \in M$ as algebraic characters X^m of the algebraic torus $T_M \cong (\mathbb{C}^*)^d$ and elements $n \in N$ as algebraic characters Y^n of the dual torus T_N. Take two families of Laurent polynomials

$$f(X) = \sum_{m \in \Delta \cap M} a_m X^m, \quad g(Y) = \sum_{n \in \Delta^\circ \cap N} b_n Y^n$$

by choosing two sufficiently general sequences of complex numbers $\{a_m\}_{m \in \Delta \cap M}$, $\{b_n\}_{n \in \Delta^\circ \cap N}$. The equations $f(X) = 0$ and $g(Y) = 0$ define two families of affine hypersurfaces $Z_f \subset T_M$ and $Z_g \subset T_N$. It is important that both families of hypersurfaces admit Calabi-Yau compactifications $\widehat{Z_f}$, $\widehat{Z_g}$ with at worst Gorenstein terminal singularities. If $d = 4$, then $\widehat{Z_f}$ and $\widehat{Z_g}$ are smooth Calabi-Yau threefolds whose Hodge numbers satisfy the equalities

$$h^{1,1}(\widehat{Z_f}) = h^{2,1}(\widehat{Z_g}), \quad h^{2,1}(\widehat{Z_f}) = h^{1,1}(\widehat{Z_g}).$$

These equalities can be considered as the first evidence of $\widehat{Z_f}$ and $\widehat{Z_g}$ being mirror symmetric.

In order to calculate the predictions for instanton numbers of rational curves on $\widehat{Z_f}$ and $\widehat{Z_g}$ one needs to know solutions of the Picard-Fuchs differential equations for their periods. It turns out that all these solutions are well-known *generalized hypergeometric functions* introduced by Gelfand, Kapranov and Zelevinsky [5] in connection with toric varieties $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^\circ}$ associated with reflexive polyhedra Δ and Δ°. Moreover, there exists a natural combinatorial way (so-called GKZ-decomposition) for describing the boundary points on the moduli spaces of $\widehat{Z_f}$ and $\widehat{Z_g}$ having the maximal unipotent monodromy. Using the generalized hypergeometric functions, one can define canonical q-coordinates in small analytic neighbourhoods of these boundary points. Using multidimensional q-expansions of so-called *normalized Yukawa couplings*, one obtains the instanton numbers of rational curves in the same way as for quintic threefolds. A detailed introduction to toric methods and their applications to mirror symmetry calculations can be found in Chapters 3-6 of the book.

3. Conclusions

As the authors observed, the greatest obstacle facing a mathematician who wants to learn about mirror symmetry is knowing where to start. Another problem is the scattering of many mathematical ideas throughout the physics literature, which is difficult for mathematicians to read. The present book seems to be a successful attempt to collect all these ideas. It could also be used as a starting reference for mathematicians interested in learning about mirror symmetry. It is especially very helpful for the reader that the authors have summarized in Appendix B some of the key points of physical theories mentioned in the book.

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