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Integral, probability, and fractal measures, by G. Edgar, Springer-Verlag, New York, 1998, x + 286 pp., $39.95, ISBN 0-387-98205-1

1. Introduction

Throughout the twentieth century mathematicians have studied the sets that are now known as "Fractals". A huge body of literature, known technically as "Geometric Measure Theory", was developed to analyse the geometric properties of these sets and related measures in the 1920's and the 1930's. However, the area received a renaissance in the late 1970's largely due to Mandelbrot's popularisation of the subject in his impressive essays [37], [38] and the advancement of computer technology allowing a wide variety of fractal sets to be drawn easily and accurately. As a natural tool for investigating the mathematics and geometry of fractals, Hausdorff measures and Hausdorff dimensions attracted new interest, and as a result a large number of textbooks (including the book under review) aimed at teaching the mathematics of fractals at an advanced undergraduate level/beginning graduate level have recently been published [2], [15], [18], [20], [21], [39], [45], [65].

Prior to the 1980's the student who wanted to learn geometric measure theory/fractal geometry had only three choices, viz. (1) original papers, (2) Rogers' excellent (and fortunately recently republished) classical text [60] on Hausdorff measures from 1970, and (3) Federer's encyclopedic and scholarly treatise [26] from 1969 containing an almost complete account of the subject prior to the mid 1960's. Unfortunately, Federer's monograph is not for the faint at heart: the author's desire to present all results in the most general setting together with his very compact way of writing makes the reading of the book an arduous, though rewarding, task. Today, however, the literature on fractals is overwhelming, ranging from the pristine Bourbakian, exemplified by Federer's scholarly treatise [26] and Falconer's excellent and popular textbooks [18], [20], to applications of fractals in many diverse areas, including among others physics, engineering, finance, biology and medicine.

Before turning towards the book under review, we will take the reader for a tour through the landscape of fractal geometry. In Section 2 we describe the basic tools of fractal geometry, viz. the box dimensions, the Hausdorff measure and the packing measure. Sections 3 and 4 give an overview of (some aspects of) fractal geometry of sets, and Section 5 describes the fractal geometry of measures. In Section 6 we discuss one of the most studied class of fractal sets and measures, viz. self-similar sets and self-similar measures. Finally, in section 7 we return to the book under review.

2. The tools: Box dimensions, the Hausdorff measure and the packing measure

2.1. Box dimensions. The box dimension is one of the most widely used fractional dimensions. The definition goes back to the late 1920's, and its popularity is largely due to its relative ease of rigorous computation and numerical estimation. Let E be a non-empty bounded subset of $\mathbb{R}^n$ (or of a totally bounded metric space).

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For $\delta > 0$, let $N_\delta(E)$ denote the smallest number of sets of diameter at most δ which can cover E. The number $N_\delta(E)$ obviously increases as δ decreases, and we might expect that $N_\delta(E)$ behaves roughly like a power of $1/\delta$; i.e. we might expect that there exists a positive number d such that

$$N_\delta(E) \sim (1/\delta)^d \quad \text{for } \delta \text{ close to } 0. \tag{2.1}$$

Indeed, if $E = [0,1]^n$ is the n-dimensional unit-cube in $\mathbb{R}^n$, then (2.1) is clearly satisfied for $d = n$. If (2.1) holds, we might therefore be tempted to interpret the number d as the dimension of E. This leads to the following definition (obtained formally by solving (2.1) for d). The lower and upper box dimensions of E are defined by

$$\underline{\dim}_{\mathsf{B}}(E) = \liminf_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta}$$

and

$$\overline{\dim}_{\mathsf{B}}(E) = \limsup_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta},$$

respectively. If $\underline{\dim}_{\mathsf{B}}(E)$ and $\overline{\dim}_{\mathsf{B}}(E)$ coincide, we denote the common value by $\dim_{\mathsf{B}}(E)$ and refer to it as the box dimension of E. The box dimension has two major advantages. Firstly, it is appealing for numerical and experimental purposes in the sciences, since it can be estimated as the slope of a log-log plot of $N_\delta(E)$ vs. δ for a suitable range of δ. Secondly, the box dimension is usually relatively easy to compute rigorously. Indeed, it is a small exercise to see that the box dimension of the middle third Cantor set equals $\frac{\log 2}{\log 3}$ (see, for example, the six line calculation in [20, p. 43]). However, although box dimensions are relatively easy to compute, they have one major drawback: they do not distinguish between a set and its closure; i.e. for $E \subseteq \mathbb{R}^n$ we have

$$\underline{\dim}_{\mathsf{B}}(E) = \underline{\dim}_{\mathsf{B}}(\overline{E}),\ \overline{\dim}_{\mathsf{B}}(E) = \overline{\dim}_{\mathsf{B}}(\overline{E}),$$

where $\overline{E}$ denotes the closure of E. It follows from this that the set $\mathbb{Q} \cap [0,1]$ of rationals in $[0,1]$ has box dimension equal to 1,

$$\dim_{\mathsf{B}}(\mathbb{Q} \cap [0,1]) = \dim_{\mathsf{B}}(\overline{\mathbb{Q} \cap [0,1]}) = \dim_{\mathsf{B}}([0,1]) = 1.$$

No-one would regard the countable set $\mathbb{Q} \cap [0,1]$ as "big", yet its box dimension equals 1. This severely limits the usefulness of box dimensions in pure mathematics. Very roughly speaking, the box counting approach to fractional dimensions is too sensitive to accumulation points and therefore often gives dimensions that are "too big". This disadvantage is modified by introducing more sophisticated measure theoretical notions of dimensions based on the Hausdorff measure and the packing measure.

2.2. Hausdorff measure and Hausdorff dimension. The Hausdorff measure provides a natural way of measuring the t-dimensional volume of a set. These ideas were introduced by Constantin Carathéodory [10] in 1914 and subsequently generalised and investigated further by Felix Hausdorff [30] in 1919. The reader will find English translations of Carathéodory's 1914 paper and Hausdorff's 1919 paper (and several other classical papers on fractals) in Edgar's anthology [16]. The modern idea of defining a measure to extend the notion of the length of an interval to more complicated sets goes back to the 1890's. Indeed, Borel used measures to estimate the size of sets in his study of "pathological" real functions in 1895; cf. [31]. These ideas were subsequently taken up by Lebesgue in the early part of this century in his construction of the Lebesgue integral, and by 1910 the "length" of rather general subsets of the real line was covered and reasonably well understood due to Lebesgue's theory. Carathédory was interested in extending and generalizing this theory to higher dimensional spaces. In particular, Carathédory was interested in the following question: what is the natural "length" of a subset E of $\mathbb{R}^n$? He solves this problem in the following simple and elegant way. For $\delta > 0$, we define the δ approximative length of E by

$$\mathcal{H}_\delta(E) = \inf \left\{ \sum_i \operatorname{diam}(E_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i\,,\ \operatorname{diam}(E_i) < \delta \right\}\ ; \tag{2.2}$$

i.e. we look at all the covers of E by sets of diameter at most δ and seek to minimize the sum of their diameters. The length (or linear measure) $\mathcal{H}(E)$ of E is now defined by

$$\mathcal{H}(E) = \sup_{\delta > 0} \mathcal{H}_\delta(E)\,. \tag{2.3}$$

Carathédory then shows that $\mathcal{H}$ is a Borel measure and that $\mathcal{H}(E)$ coincides with our intuitive notion of length: if γ is a curve without intersections, i.e. a single valued continuous image of the unit interval, then $\mathcal{H}(\gamma)$ equals the supremum of the lengths of polygons inscribed in γ. Finally, Carathéodory remarks that the construction in (2.2) and (2.3) can be generalized to give an m-dimensional measure in $\mathbb{R}^n$ for any positive integer m with $m < n$ simply by replacing the sum $\sum_i \operatorname{diam}(E_i)$ in (2.2) by

$$\sum_i \operatorname{diam}_m(E_i)\,,$$

where $\operatorname{diam}_m(E_i)$ denotes the supremum of the m-dimensional volumes of all orthogonal projections of the convex hull of E_i onto all m-dimensional subspaces of $\mathbb{R}^n$.

This prompted Hausdorff's seminal 1919 paper "Dimension und äußeres Maß" [30] based on the far-reaching observation that Carathéodory's construction makes sense (and is useful) even if m is not an integer. Hausdorff begins by paying tribute to Carathéodory's 1914 paper. He then modestly writes, "Hierzu geben wir in folgende eine kleinen Beitrag." This "kleinen Beitrag" is nothing less than the entire theory of fractional dimensions. In Hausdorff's extension to the m-dimensional case, the m-dimensional measure is based on the sum,

$$\sum_i \operatorname{diam}(E_i)^m\,,$$

of the diameters of the sets E_i to the m-th power. In this formulation the various formulae still make sense even when the integral dimension m is replaced by an *arbitrary* positive real number $t \geq 0$ (as opposed to Carathéodory, who only considered integer values of $t = m$). Let X be a metric space (Hausdorff worked in $\mathbb{R}^n$), and let $t \geq 0$ be an *arbitrary* positive real number. Fix a subset $E \subseteq X$ and $\delta > 0$. The δ approximative t-dimensional Hausdorff measure $\mathcal{H}^t_\delta(E)$ of E is defined by

$$\mathcal{H}^t_\delta(E) = \inf \left\{ \sum_i \operatorname{diam}(E_i)^t \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i\,,\ \operatorname{diam}(E_i) < \delta \right\}\ ,$$

and the t-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of E is defined by

$$\mathcal{H}^t(E) = \sup_{\delta>0} \mathcal{H}^t_\delta(E)\,.$$

It is easily seen that each set E has a unique "correct" value of the dimension parameter t associated with it; i.e. there exists a unique value, $\dim(E)$, of t for which the Hausdorff measure $\mathcal{H}^t(E)$ "drops" from infinity to zero, i.e.

$$\mathcal{H}^t(E) = \begin{cases} \infty & \text{for } t < \dim(E); \\ 0 & \text{for } \dim(E) < t. \end{cases}$$

The number $\dim(E)$ is the Hausdorff dimension of E. We emphasize that the measure, $\mathcal{H}^{\dim(E)}(E)$, at the critical dimension need not be positive and finite; there are many examples for which $\mathcal{H}^{\dim(E)}(E)$ equals either zero or infinity (see below). Let us list a few of the main properties of the Hausdorff measure and the Hausdorff dimension:

- The scaling law of Hausdorff measures in Euclidean spaces: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with similarity ratio r (i.e. $|f(x) - f(y)| = r|x-y|$ for all $x, y \in \mathbb{R}^n$), then $\mathcal{H}^t(f(E)) = r^t\mathcal{H}^t(E)$ for all $E \subseteq \mathbb{R}^n$;
- The Hausdorff dimension is monotone: if $E \subseteq F$, then $\dim(E) \le \dim(F)$;
- The Hausdorff dimension is countably stable: $\dim(\cup_{n\in\mathbb{N}}E_n) = \sup_{n\in\mathbb{N}}\dim(E_n)$;
- Countable subsets of $\mathbb{R}^n$ have zero dimension: if E is a countable subset of $\mathbb{R}^n$, then $\dim(E) = 0$;
- Open subsets of $\mathbb{R}^n$ have the correct dimension: if G is an open and non-empty subset of $\mathbb{R}^n$, then $\dim(G) = n$;
- In n-dimensional Euclidean space $\mathbb{R}^n$, the n-dimensional Hausdorff measure is proportional to the n-dimensional Lebesgue measure.

As an example, we will (almost) prove that the Hausdorff dimension of the middle-third Cantor set C is $\frac{\log 2}{\log 3}$. In fact, Hausdorff proved this in his paper. Consider the two similarities $f_1(x) = \frac{x}{3}$ and $f_2(x) = \frac{x}{3} + \frac{2}{3}$. Then we have $C = f_1(C) \cup f_2(C)$. Since $f_1(C)$ and $f_2(C)$ are disjoint and $\mathcal{H}^t$ is a measure, we now obtain

$$\mathcal{H}^t(C) = \mathcal{H}^t(f_1(C)) + \mathcal{H}^t(f_2(C))\,.$$

By the scaling law, $\mathcal{H}^t(f_1(C)) = (\frac{1}{3})^t\mathcal{H}^t(C)$ and similarly $\mathcal{H}^t(f_2(C)) = (\frac{1}{3})^t\mathcal{H}^t(C)$. Hence

$$\mathcal{H}^t(C) = (\tfrac{1}{3})^t\mathcal{H}^t(C) + (\tfrac{1}{3})^t\mathcal{H}^t(C) = 2(\tfrac{1}{3})^t\mathcal{H}^t(C)\,.$$

And so

$$1 = 2(\tfrac{1}{3})^t\,,$$

i.e. $t = \frac{\log 2}{\log 3}$. Hence, we thus deduce that the Hausdorff dimension of C equals $\frac{\log 2}{\log 3}$. Of course, this is only a heuristic argument, since we cannot cancel $\mathcal{H}^t(C)$ unless $\mathcal{H}^t(C)$ is positive and finite, but this can be justified. In fact, it can be proved that if $t = \dim(C) = \frac{\log 2}{\log 3}$, then $\mathcal{H}^t(C) = 1$.

In the previous example, the Hausdorff measure evaluated at the critical dimension was positive and finite. However, often this is not the case. In those cases it is useful to introduce a finer notion of dimension. A right continuous function

$h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ is called a dimension function (or a gauge function). For a dimension function h we define the h-dimensional Hausdorff measure $\mathcal{H}^h$ as follows. For $E \subseteq X$ and $\delta > 0$ write

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_i h(\mathrm{diam}(E_i)) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i \,,\ \mathrm{diam}(E_i) < \delta \right\} .$$

The h-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of E is then defined by

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}^t_\delta(E) \,.$$

Observe that if $t \geq 0$ and h_t denotes the power function $h_t(r) = r^t$, then $\mathcal{H}^{h_t} = \mathcal{H}^t$. We will now say that h is the exact dimension function of a set E if

$$0 < \mathcal{H}^h(E) < \infty \,.$$

Hence, the power function $h(r) = r^t$ with $t = \frac{\log 2}{\log 3}$ is the exact dimension function of the middle-third Cantor set. However, for other naturally occurring sets, the exact power function often involves logarithmic corrections. For example, with probability 1, a Brownian path X in $\mathbb{R}^n$ (with $n \geq 2$) has Hausdorff dimension equal to 2, but the 2-dimensional Hausdorff measure $\mathcal{H}^2(X)$ is zero. In this case the exact dimension function is given by

$$h(r) = \begin{cases} r^2 \log \frac{1}{r} \log\log\log \frac{1}{r} & \text{for } n = 2; \\ r^2 \log\log \frac{1}{r} & \text{for } n \geq 3; \end{cases}$$

i.e. with probability 1, we have $0 < \mathcal{H}^h(X) < \infty$ for a Brownian path X in $\mathbb{R}^n$. Hence, the Hausdorff dimension of a Brownian path is "logarithmically smaller" than 2. The best introduction to the theory of general h-dimensional Hausdorff measures is undoubtedly Rogers' excellent classical textbook [60].

2.3. Packing measure and packing dimension. The packing measure was introduced by Tricot [64], Taylor & Tricot [63] and Raymond & Tricot [59] in the mid 1980's as a dual to the Hausdorff measure: the Hausdorff measure is defined by considering economical coverings, whereas the packing measure is defined by considering efficient packings. The packing measure is nowadays considered as important as the Hausdorff measure. Many Hausdorff measure properties have dual packing measure properties, and dual Hausdorff measure and packing measure results are usually presented alongside each other. We will now define the packing measure. Let X be a metric space, $E \subseteq X$ and $\delta > 0$. A countable family $(B(x_i, r_i))_i$ of closed balls in X is called a centered δ-packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ for all $i \neq j$. Write

$$\overline{\mathcal{P}}^t_\delta(E) = \sup \left\{ \sum_i (2r_i)^t \,\middle|\, (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\} ;$$

i.e. we look at all centered packings of E by balls of radii at most δ and seek to maximize the sum of twice their radii to the t-th power. The t-dimensional prepacking measure $\overline{\mathcal{P}}^t(E)$ of E is now defined by

$$\overline{\mathcal{P}}^t(E) = \inf_{\delta > 0} \overline{\mathcal{P}}^t_\delta(E) \,.$$

However, the set function $\overline{\mathcal{P}}^t$ is not necessarily countably subadditive, and hence not necessarily an (outer) measure; cf. [63]. But $\overline{\mathcal{P}}^t$ gives rise to a Borel measure, namely the t-dimensional packing measure $\mathcal{P}^t(E)$ of E, as follows:

$$\mathcal{P}^t(E) = \inf_{E \subseteq \cup_{i=1}^{\infty} E_i} \sum_i \overline{\mathcal{P}}^t(E_i) \,.$$

The packing measure also gives rise to a dimension. For a subset E of X the packing dimension, $\mathrm{Dim}(E)$, of E is defined analogously to the Hausdorff dimension, i.e. as the unique value of t for which the packing measure of E "drops" from infinity to zero,

$$\mathcal{P}^t(E) = \begin{cases} \infty & \text{for } t < \mathrm{Dim}(E); \\ 0 & \text{for } \mathrm{Dim}(E) < t. \end{cases}$$

The packing measure and the packing dimension have properties similar to those enjoyed by the Hausdorff measure and the Hausdorff dimension and listed in Section 2.2. Of course, we can also define a h-dimensional packing measure $\mathcal{P}^h$ for an arbitrary dimension function h analogously to the h-dimensional Hausdorff measure $\mathcal{H}^h$.

There are two main reasons for the importance of the packing measure. Firstly, the packing measure has (in many cases) properties dual to those of the Hausdorff measure thereby complementing and providing further insight into already existing results. For example, the Hausdorff dimension and the packing dimension behave dually under the formation of Cartesian products: if $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$, then

$$\begin{aligned} \dim(E) + \dim(F) &\leq \dim(E \times F) \\ &\leq \dim(E) + \mathrm{Dim}(F) \\ &\leq \mathrm{Dim}(E \times F) \\ &\leq \mathrm{Dim}(E) + \mathrm{Dim}(F) \,. \end{aligned} \tag{2.4}$$

Secondly, recall from Section 2.2 that there exist sets E such that the Hausdorff measure $\mathcal{H}^d(E)$ of E at the critical dimension $d = \dim(E)$ equals zero, i.e. $\mathcal{H}^d(E) = 0$. Hence, in those cases the Hausdorff measure is not the "correct" d-dimensional measure on E. However, for such sets the packing measure may be the "correct" d-dimensional measure; i.e. $\mathcal{P}^d(E)$ may be positive and finite. Indeed, consider the set $J \subseteq \mathbb{C}$ of simple complex continued fractions whose partial denominators are Gaussian integers of the form $b = m + in$ with $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, i.e.

$$J = \left\{ \frac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \;\middle|\; b_k = m_k + in_k \,,\; m_k \in \mathbb{N} \,,\; n_k \in \mathbb{Z} \right\} .$$

For this set the Hausdorff dimension and packing dimension coincide, i.e. $\dim(J) = \mathrm{Dim}(J) := d$; cf [49]. However, the d-dimensional Hausdorff measure of J equals zero,

$$\mathcal{H}^d(J) = 0 \,,$$

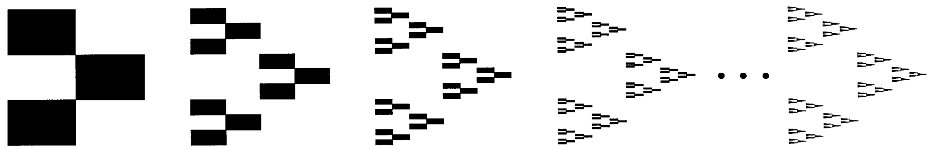

FIGURE 1. The construction of the self-affine set K.

whereas the d-dimensional packing measure of J is positive and finite,

$$0 < \mathcal{P}^d(J) < \infty .$$

Hence, in this case the "correct" d-dimensional measure supported by J is the d-dimensional packing measure and not the d-dimensional Hausdorff measure. There are also examples of sets E for which the Hausdorff measure at the critical dimension is positive and finite, whereas the packing measure is infinite.

It is natural to ask if there is any relationship between the box dimensions, the Hausdorff dimension and the packing dimension. Tricot proved that the Hausdorff measure is always less than the packing measure, i.e.

$$\mathcal{H}^t(E) \leq \mathcal{P}^t(E)$$

for all $t \geq 0$ and all $E \subseteq X$; in particular, it follows that $\dim(E) \leq \mathrm{Dim}(E)$ for all $E \subseteq X$. More generally the following inequalities are always satisfied:

$$\dim(E) \leq \mathrm{Dim}(E) \leq \overline{\dim}_{\mathsf{B}}(E) ,$$
$$\dim(E) \leq \underline{\dim}_{\mathsf{B}}(E) \leq \overline{\dim}_{\mathsf{B}}(E) ,$$

for all $E \subseteq \mathbb{R}^n$. All of the above inqualities can be strict. A nice example showing that the inequality between the Hausdorff dimension and the packing dimension can be strict is provided by the so-called self-affine set K shown in Figure 1. The set K in Figure 1 has the following dimensions:

$$\dim(K) = \log\left(2^{\log 2/\log 3} + 1\right) / \log 2 \approx 1.3497 ,$$
$$\mathrm{Dim}(K) = 1 + \log(3/2)/\log 3 \approx 1.3691 .$$

The Hausdorff dimension of K was obtained independently by Bedford [3] and the 1998 Fields medalist Curtis McMullen [46] in 1984. The value for the packing dimension of K was obtained by Peres [54] in 1994.

3. The geometry of fractal sets. Local properties

3.1. Densities. The early work on the local geometric structure of sets in Euclidean space was pioneered by A. Besicovitch in the 1920's and the 1930's in a long series of papers including the three fundamental papers [4], [5], [6]. It is natural to ask how "dense" a set E is in a neighbourhood of a point x. The classical result, Lebesgue's Density Theorem, says that at almost all points of E the set is as "dense" as possible: if E is a Borel set in $\mathbb{R}^n$, then the density

$$D(E, x) = \lim_{r \searrow 0} \frac{\mathcal{L}^n(E \cap B(x,r))}{\mathcal{L}^n(B(x,r))} ,$$

where $\mathcal{L}^n$ denotes n-dimensional Lebesgue measure in $\mathbb{R}^n$, equals 1 for almost all x in E (and equals 0 for almost all x in the complement of E); i.e.

$$D(E,x) = 1 \quad \text{for } \mathcal{L}^n \text{ almost all } x \in E. \tag{3.1}$$

Hence, if $\dim(E) < n$ and thus $\mathcal{L}^n(E) = 0$, Lebesgue Density Theorem becomes trivial. In this case it is natural to investigate the s-dimensional density of E for $s = \dim(E)$ by comparing the amount of s-dimensional mass $\mathcal{H}^s(E \cap B(x,r))$ of E in the ball $B(x,r)$ with r^s. For $s > 0$ and $x \in \mathbb{R}^n$ we therefore define the lower and upper s-dimensional densities, $\underline{D}^s(E,x)$ and $\overline{D}^s(E,x)$, of E at x by

$$\underline{D}^s(E,x) = \liminf_{r \searrow 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s},$$
$$\overline{D}^s(E,x) = \limsup_{r \searrow 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s}.$$

If $\underline{D}^s(E,x) = \overline{D}^s(E,x)$, we refer to the common value as the density of E at x and denote it by $D^s(E,x)$. We can now formulate the classical density theorem relating Hausdorff measures and upper densities due to Besicovitch [4], [5]: if $E \subseteq \mathbb{R}^n$ is an s-set, i.e. $0 < \mathcal{H}^s(E) < \infty$, then

$$2^{-s} \le \overline{D}^s(E,x) \le 1 \quad \text{for } \mathcal{H}^s \text{ almost all } x \in E. \tag{3.2}$$

There is a dual density theorem for packing measures and lower densities due to Raymond & Tricot [59] from 1988: if $E \subseteq \mathbb{R}^n$ is a packing s-set, i.e. $0 < \mathcal{P}^s(E) < \infty$, then

$$\underline{D}^s(E,x) = 1 \quad \text{for } \mathcal{P}^s \text{ almost all } x \in E. \tag{3.3}$$

3.2. Regular sets and rectifiability. From Besicovitch to Preiss. There is one important difference between Lebesgue's Density Theorem (3.1) and (3.2) and (3.3). For $s = n$, Lebesgue's Density Theorem guarantees that the density exists (and equals 1) for almost all $x \in E$, but for $0 < s < n$, the density might fail to exist. For example, the middle-third Cantor set C is an s-set for $s = \frac{\log 2}{\log 3}$ with $\underline{D}^s(C,x) < \overline{D}^s(C,x)$ for all $x \in C$. This leads to the important notion of regular and irregular points and sets. Let E be an s-set of $\mathbb{R}^n$. A point $x \in E$ is called a regular point if $D^s(E,x) = 1$, and irregular otherwise. The set E is called regular if $\mathcal{H}^s$ almost all points of E are regular, and E is called irregular if $\mathcal{H}^s$ almost all points of E are irregular. The characterization of regular and irregular sets by their geometric properties such as the existence of (measure theoretically defined) tangents and m-rectifiability (a set E is called m-rectifiable for $m = 1, 2, \dots, n-1$ if it is made up of an $\mathcal{H}^m$ null-set and countably many pieces which are Lipschitz images of $\mathbb{R}^m$, i.e. if there exist countably many Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that $\mathcal{H}^m(E \setminus \cup_i f_i(\mathbb{R}^m)) = 0$) was initiated by Besicovitch and is still flourishing today. In particular, Besicovitch [4], [5] proved that a 1-set in the plane is regular if and only if it is 1-rectifiable. However, due to the significant amount of technical difficulties it took almost half a century before these results were extended to higher dimension by Marstrand [40], [42], [43] in the 1960's and finally by Mattila [44] in 1974. The text by Falconer [18] contains a nice introduction to this subject with complete proofs for the planar case, and the texts by Ferderer [26] and Mattila [45] contain thorough and detailed discussions of the general case. The study of the relationship between densities, rectifiability and existence of tangents culminated in 1987 with

Preiss' formidable paper [56] in which he proves that if the s-dimensional density $\lim_{r\searrow 0}\mu(B(x,r))/(2r)^s$ of an arbitrary Radon measure μ exists and is positive and finite for μ almost all x, then s is an integer and μ is s-rectifiable (i.e. there exists an s-rectifiable set E such that $\mu(\mathbb{R}^n\setminus E)=0$ and $\mu\ll\mathcal{H}^s$). The classical result regarding the equivalence between regularity and rectifiability of an s-set E due to Marstrand and Besicovitch can readily be obtained from Preiss' theorem by applying it to the measure μ defined by $\mu(B)=\mathcal{H}^s(E\cap B)$. In order to achieve this result Preiss introduces and uses in a very powerful way the notion of tangent measures. Tangent measures describe the local behaviour of a measure μ in a neighbourhood of a given point a but behave in a more regular way. For $a\in\mathbb{R}^n$ and $r>0$ define the measure $T_{a,r}\mu$ by

$$T_{a,r}\mu(E)=\mu(rE+a)\,;$$

i.e. the operator $T_{a,r}$ "blows up" the original measure μ by the factor $1/r$ and shifts it by the amount a. A non-zero Radon measure ν is called a tangent measure of μ at a if there exist two sequences $(r_n)_n$ and $(c_n)_n$ of positive reals with $r_n\to 0$ such that

$$c_nT_{a,r_n}\mu\to\nu\quad\text{weakly as } n\to\infty.$$

Tangent measures have become one of the most important technical tools in contemporary geometric measure theory and have played a major part in the study of the connection between singular integrals and rectifiability.

Even though the density of an irregular set fails to exist at almost all points, the so-called average density does exist almost everywhere for many irregular sets. If the s-dimensional density of a set E fails to exist at a point x, then the ratio $\mathcal{H}^s(E\cap B(x,r))/(2r)^s$ "oscillates" between the lower and upper density for small r, and it is therefore natural to try and describe this "oscillation" by considering the "average" density. This approach was developed by Bedford & Fisher [7] in 1992. They define the s-dimensional lower and upper average densities, $\underline{A}^s(E,x)$ and $\overline{A}^s(E,x)$, by replacing r with e^{-t} and applying the Cesaro average to $\mathcal{H}^s(E\cap B(x,e^{-t}))/(2e^{-t})^s$, i.e.

$$\underline{A}^s(E,x)=\liminf_{T\to\infty}\frac{1}{T}\int_0^T\frac{\mathcal{H}^s(E\cap B(x,e^{-t}))}{(2e^{-t})^s}\,dt\,,$$
$$\overline{A}^s(E,x)=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\frac{\mathcal{H}^s(E\cap B(x,e^{-t}))}{(2e^{-t})^s}\,dt\,.$$

They then show that the average density exists almost everywhere for Cantor sets (including the middle third Cantor set) and the zero set of a Brownian path. Average densities have recently been investigated further by a large number of authors.

4. The geometry of fractal sets. Global properties

In the previous section we described local geometrical properties of sets, i.e. properties that only depend on the behaviour of the set in a small neighbourhood of a point. However, often geometric constructions used to produce new fractal sets from old ones are "non-local". For example, the geometry of the Cartesian product of two sets or the geometry of the projection of a set onto a lower dimensional subspace depend in an essentially "non-local" way on the geometry of the original sets. In this case it is of interest to try to relate the global dimensional properties of the Cartesian product of two sets or of the projection of a set to that of the original sets.

4.1. Cartesian products. If $H \subseteq \mathbb{R}^{n+m}$ is a Borel set, then Fubini's theorem tells us that the $(n+m)$-dimensional Lebesgue measure of H can be obtained by adding up the n-dimensional Lebesgue measure of all horizontal n-dimensional sections of H; i.e.

$$\int \mathcal{L}^n(H^y)\, d\mathcal{L}^m(y) = \mathcal{L}^{n+m}(H)$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and $H^y = \{x \in \mathbb{R}^n \mid (x,y) \in H\}$ is the horizontal section of H at level $y \in \mathbb{R}^m$. Unfortunately, this result is not true for fractional dimensional Hausdorff measures or packing measures. Indeed, a classical result due to Besicivitch & Moran [8] shows that there exist sets $A, B \subseteq \mathbb{R}$ such that $\mathcal{H}^1(A \times B) > 0 = \mathcal{H}^s(A) + \mathcal{H}^t(B)$ for all $s, t > 0$ with $s + t = 1$. However, the following four Fubini type inequalities are always satisfied. For $s, t \geq 0$ there exists a constant $c > 0$ such that

$$\int \mathcal{H}^s(H^y)\, d\mathcal{H}^t(y) \leq c\mathcal{H}^{s+t}(H) \tag{4.1}$$

$$\mathcal{H}^{s+t}(E \times F) \leq c\mathcal{H}^s(E)\mathcal{P}^t(F) \tag{4.2}$$

$$\int \mathcal{H}^s(H^y)\, d\mathcal{P}^t(y) \leq c\mathcal{P}^{s+t}(H) \tag{4.3}$$

$$\mathcal{P}^{s+t}(E \times F) \leq c\mathcal{P}^s(E)\mathcal{P}^t(F) \tag{4.4}$$

for all $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$ and $H \subseteq \mathbb{R}^{n+m}$. Inequality (4.1) goes back to Besicovitch & Moran [8] and Marstrand [41]. The other inequalities are from the 1990's [29], [33]. By letting $H = E \times F$ in (4.1)–(4.4), we obtain inequalities (2.4) relating the Hausdorff dimension and the packing dimension of subsets of Euclidean spaces. The reader will observe the dual role of the Hausdorff measure and the packing measure in (4.1)–(4.4).

4.2. Projections. For a subspace Π of $\mathbb{R}^n$ we let P_Π denote the projection onto Π. Let $E \subseteq \mathbb{R}^n$ be a Borel set. For almost all m-dimensional subspaces Π of $\mathbb{R}^n$ we have

$$\dim\big(P_\Pi(E)\big) = \min\big(\dim(E), m\big)\,. \tag{4.5}$$

This result was first obtained in the planar case by Marstrand [40] in 1954 and later generalized to arbitrary dimensions by Mattila. Unfortunately, the packing dimension of the projection of a set Borel set does not satisfy a similar formula. In view of the dual role of the Hausdorff measure/Hausdorff dimension and the packing measure/packing dimension displayed in (2.4), (3.2)–(3.3) and (4.1)–(4.4) this may seem rather surprising. Indeed there is no simple formula for the packing dimension of the projection of a Borel set. Falconer & Howroyd [22] proved that

$$\frac{\mathrm{Dim}(E)}{1 + (\frac{1}{m} - \frac{1}{n})\,\mathrm{Dim}(E)} \leq \mathrm{Dim}\big(P_\Pi(E)\big) \leq \min\big(\mathrm{Dim}(E), m\big) \tag{4.6}$$

for almost all m-dimensional subspaces Π and that this inequality is the best possible. Despite, or perhaps inspite of, the complicated behaviour of packing dimensions of projected sets, this topic has recently been investigated further by a number of authors including Falconer, Howroyd and Mattila; see [22], [23], [24] for further references.

4.3. Intersections. Intersections of fractals have also been investigated. If $E \subseteq \mathbb{R}^n$ is an s-set, then

$$\begin{aligned} \dim\big(E \cap (x+\Pi)\big) &= \dim(E) - (n-m) \quad \text{if } \dim(E) = s > n-m, \\ E \cap (x+\Pi) &= \varnothing \quad \text{if } \dim(E) = s < n-m, \end{aligned} \tag{4.7}$$

for $\mathcal{H}^s$ almost all $x \in E$ and almost all m-dimensional subspaces Π [40]. More general results regarding the Hausdorff dimension of the intersection of two arbitrary subsets of $\mathbb{R}^n$ have also been obtained. As with projections, there is no simple formula for the packing dimension of the intersection of a set with an affine subspace; see [24], [25].

5. Fractal measures

5.1. Dimensions of measures. Measures have always been a fundamental tool in the study of the geometry of fractal sets. Indeed, Section 3 showed many instances where the geometric properties of a set are analysed by studying properties of measures supported by the set. Because of this and due to the existence of natural fractal measures in many constructions in fractal geometry and dynamical systems (for example, so-called self-similar measures (see below) and the occupation measure on an attractor of a chaotic dynamical system), fractal properties of measures have been studied intensively in their own right during the past 10-15 years. A natural measure of the "fractalness" or "singularity" of a measure is provided by "the size of the smallest set that supports the measure". This idea leads to the notion of the Hausdorff dimensions and packing dimensions of the measure. For a measure μ on $\mathbb{R}^n$ the lower and upper Hausdorff dimensions and packing dimensions of μ are defined by

$$\begin{aligned} \underline{\dim}(\mu) &= \inf\{\dim(E) \mid \mu(E) > 0\}\,, \\ \overline{\dim}(\mu) &= \inf\{\dim(E) \mid \mu(\mathbb{R}^n \setminus E) = 0\}\,, \\ \underline{\mathrm{Dim}}(\mu) &= \inf\{\mathrm{Dim}(E) \mid \mu(E) > 0\}\,, \\ \overline{\mathrm{Dim}}(\mu) &= \inf\{\mathrm{Dim}(E) \mid \mu(\mathbb{R}^n \setminus E) = 0\}\,. \end{aligned}$$

One of the main reasons for the importance of the Hausdorff dimensions and packing dimensions of a measure is their close connection with densities; in fact,

$$\underline{\dim}(\mu) = \operatorname{ess\,inf}_x \ \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}\,, \tag{5.1}$$

$$\overline{\dim}(\mu) = \operatorname{ess\,sup}_x \ \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}\,, \tag{5.2}$$

$$\underline{\mathrm{Dim}}(\mu) = \operatorname{ess\,inf}_x \ \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}\,, \tag{5.3}$$

$$\overline{\mathrm{Dim}}(\mu) = \operatorname{ess\,sup}_x \ \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}\,, \tag{5.4}$$

where ess inf and ess sup denote the essential infimum and the essential supremum with respect to μ. Again the reader will observe the dual role of the Hausdorff dimension and the packing dimension in (5.1)–(5.4). Cutler [12], [13] has developed a theory of "dimensional decompositions" of measures based on (5.1)–(5.4). Measure theoretical analogues of (2.4), (4.5), (4.6) and (4.7) for product measures, projections of measures and intersections of measures with affine subspaces have been investigated [24], [29]. Cutler, Pesin, Young and others [14], [55], [66] have proved that many measures μ that occur naturally in fractal geometry and dynamical systems (including self-similar measures, Gibbs' states and ergodic measures invariant under sufficiently smooth maps) are d-uni-dimensional; i.e.

$$\underline{\dim}(\mu) = \overline{\dim}(\mu) = \underline{\mathrm{Dim}}(\mu) = \overline{\mathrm{Dim}}(\mu) = d\,,$$

or equivalently $\lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} = d$ for μ almost all x.

5.2. Multifractals. Multifractal analysis, which has become one of the most fashionable topics in geometric measure theory and dynamical systems in the 1990's, studies the local structure of measures with "widely varying intensity" and provides a finer analysis than the notion of d-uni-dimensionality described in Section 5.1. Even though a measure μ is d-uni-dimensional, i.e. $\lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} = d$ for μ almost all x, the set $\Delta(\alpha)$ of those points x for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with center x and radius r behaves like r^α for small r; i.e.

$$\Delta(\alpha) = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}$$

may still be substantial for a range of values of α. If this is the case, then the measure is called a multifractal measure or simply a multifractal, and it is natural to study the sizes of the sets $\Delta(\alpha)$ as α varies. We therefore study the Hausdorff dimension,

$$f(\alpha) = \dim \Delta(\alpha)\,, \tag{5.5}$$

or the packing dimension,

$$F(\alpha) = \mathrm{Dim}\, \Delta(\alpha)\,, \tag{5.6}$$

of the sets $\Delta(\alpha)$ as a function of α. The functions in (5.5) and (5.6) and similar functions are generically known as "the multifractal spectrum of μ", and one of the main problems in multifractal analysis is to study these and related functions. The function $f(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [32]. Based on a remarkable insight together with a clever heuristic argument Halsey et al. [32] suggest that the multifractal spectrum $f(\alpha)$ can be computed in the following way, known as the so-called Multifractal Formalism in the physics literature.

The Multifractal Formalism – A Physics Folklore Theorem.

(i) *For each $q \in \mathbb{R}$ the following limit exists:*

$$\tau(q) = \lim_{r\searrow 0} \frac{\log\left(\sup \sum_i \mu(U_i)^q\right)}{-\log r}$$

where the supremum is over all countable partitions $(U_i)_i$ of the support of μ with $\sup_i \mathrm{diam}(U_i) \le r$.

(ii) *The multifractal spectrum of μ equals the Legendre transform of τ,*

$$f(\alpha) = \begin{cases} \tau^*(\alpha) & \text{if } 0 \le \tau^*(\alpha); \\ 0 & \text{if } \tau^*(\alpha) < 0, \end{cases}$$

where τ^* denotes the Legendre transform of τ, i.e. $\tau^*(\alpha) = \inf_q(\alpha q + \tau(q))$.

Very recently there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectrum of various measures in the mathematical literature, and within the last 3 or 4 years the multifractal spectra of various classes of measures exhibiting some degree of self-similarity have been computed rigorously; cf. for example [11], [17], [52], [53], [55], [9].

6. Self-similar sets and self-similar measures

Self-similar sets and self-similar measures have become one of the most studied classes of sets and measures in the 1980's and 1990's largely because they are sufficiently well-behaved to admit a detailed study and because a large number of existing "classical" fractals and fractal measures, such as the von Koch curve and the Sierpinski triangle, fit into the framework of self-similarity.

6.1. Self-similar sets. A set is called self-similar if it is built up of scaled down pieces that are geometrically similar to the entire set. For example, if C denotes the middle-third Cantor set, then $C = C_1 \cup C_2$ where $C_1 = C \cap [0, \frac{1}{3}]$ and $C_2 = C \cap [\frac{2}{3}, 1]$ are geometrically similar copies of C each scaled down by the factor $\frac{1}{3}$.

The formal characterization and study of self-similar sets has been one of the main fields of interest in fractal geometry in the 1980's and 1990's. The formal definition and initial investigation of self-similar sets goes back to Hutchinson's elegant treatment in 1981 [34]. Let X be a complete metric space, and let $S_1, \dots, S_N : X \to X$ be contractive mappings. Then there exists a unique non-empty compact subset K of X such that

$$K = \bigcup_{i=1}^{N} S_i(K)\,. \tag{6.1}$$

The list $(X, S_1, \dots, S_N)$ is (using a terminology introduced and popularized by Barnsley [2]) called an iterated function system (IFS), and the set K is called the invariant set or self-similar set associated with the IFS. The existence and uniqueness of the invariant set K is easily established by the following elegant argument due to Hutchinson. Let $\mathcal{K}(X)$ denote the family of all compact and non-empty subsets of X, and equip $\mathcal{K}(X)$ with the Hausdorff metric d. Define $S : \mathcal{K}(X) \to \mathcal{K}(X)$ by $S(M) = \cup_i S_i(M)$. Then S is a contraction with respect to d, and since $(\mathcal{K}(X), d)$ is a complete metric space, Banach's contraction mapping theorem shows that S has a unique fixed point $K \in \mathcal{K}(X)$, i.e. $K = S(K) = \cup_i S_i(K)$.

If $X = \mathbb{R}^n$ and each S_i is a similarity with ratio r_i, i.e. $|S_i(x) - S_i(y)| = r_i|x - y|$ for all $x, y \in \mathbb{R}^n$, then (6.1) shows that K is made up of N pieces, $S_1(K), \dots, S_N(K)$, which are geometrically similar to the entire set K but scaled by the factors $r_1, \dots, r_N$ respectively. If in addition the copies $S_1(K), \dots, S_N(K)$ are disjoint or "nearly disjoint", i.e. satisfying the celebrated Open Set Condition from [34], then the Hausdorff dimension and the packing dimension of K can be found [34]: the Hausdorff dimension $\dim(K)$ and the packing dimension $\mathrm{Dim}(K)$ of K equal the unique positive number s satisfying

$$\sum_{i=1}^{N} r_i^s = 1\,, \tag{6.2}$$

and furthermore, the Hausdorff measure and the packing measure of K at the critical dimension s are both positive and finite, i.e. $0 < \mathcal{H}^s(K) \le \mathcal{P}^s(K) <$

∞. In fact, formula (6.2) had already been established in certain cases by Moran [47] 35 years earlier in 1946. The IFS approach has now been extended to so-called graph-directed constructions by Mauldin & Williams [51]; to families with an infinite number of contracting similarities by Mauldin & Urbanski [49]; to families of random contractive similarities by Falconer [19], Graf [28] and Mauldin & Williams [50]; and to families of non-linear contractions by Pesin [55].

6.2. Self-similar measures. The notion of self-similarity can be extended to measures. Intuitively a measure is called self-similar if it is built up of translated and scaled down pieces of itself. Formally we proceed as follows. Let $(X, S_1, \dots, S_N)$ be an IFS with invariant set K, and let $(p_1, \dots, p_N)$ be a probability vector. Then there exists a unique probability measure μ such that

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}\,, \tag{6.3}$$

where $\mu \circ S_i^{-1}$ denotes the image measure of μ under S_i, i.e. $(\mu \circ S_i^{-1})(B) = \mu(S_i^{-1}(B))$. The list $(X, S_1, \dots, S_N, p_1, \dots, p_N)$ is called an iterated function system with probabilities (IFSP), and the measure μ is called the invariant measure or self-similar measure associated with the IFSP.

If $X = \mathbb{R}^n$ and each S_i is a similarity with ratio r_i, then (6.3) shows that μ is made up of N measures, $p_1\mu \circ S_1^{-1}, \dots, p_N\mu \circ S_N^{-1}$, which are translated copies of the original measure μ but scaled by the factors $p_1, \dots, p_N$ respectively. Self-similar measures have been the focus for much study in the 1990's. For example, if the IFS satisfies the Open Set Condition, then the dimensions of μ (see Section 5.1) can be found [27]:

$$\underline{\dim}(\mu) = \overline{\dim}(\mu) = \underline{\mathrm{Dim}}(\mu) = \overline{\mathrm{Dim}}(\mu) = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log r_i}\,.$$

The multifractal spectra of μ (see Section 5.2) can also be found [1], [11]: define the function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\sum_{i=1}^{N} p_i^q r_i^{\beta(q)} = 1\,; \tag{6.4}$$

then the following limit exists for all q and equals $\beta(q)$:

$$\beta(q) = \lim_{r \searrow 0} \frac{\log\left(\sup \sum_i \mu(U_i)^q\right)}{-\log r}\,, \tag{6.5}$$

where the supremum is over all countable partitions $(U_i)_i$ of the support of μ with $\sup_i \operatorname{diam}(U_i) \le r$, and the multifractal spectra of μ (see (5.5) and (5.6)) equal the Legendre transform β^* of β,

$$f(\alpha) = F(\alpha) = \begin{cases} \beta^*(\alpha) & \text{if } 0 \le \beta^*(\alpha); \\ 0 & \text{if } \beta^*(\alpha) < 0. \end{cases} \tag{6.6}$$

As with self-similar sets, the IFSP approach to measures has been extended to so-called graph-directed constructions, to families of random contractive similarities and to families of non-linear contractions; see [1], [9], [17], [53], [55] for further references.

Self-similar sets and measures for which the sets $S_1(K), \dots, S_N(K)$ are "nearly disjoint" are well understood by now. Recently there has been an enormous interest in investigating self-similar constructions in which the sets $S_1(K), \dots, S_N(K)$ may have considerable overlap. Because of the overlap between the sets $S_1(K), \dots, S_N(K)$, this problem is extremely difficult. However, despite the difficulties encountered, a number of deep results have been obtained; see [35], [36], [57], [58], [61], [62].

7. The book

This is a nice and well written textbook in geometric measure theory/fractal geometry at the beginning graduate level. The book is intended as a continuation of the author's undergraduate textbook in fractal geometry [15].

The book consists of 5 chapters. Chapter 1 ("Fractal Measures") covers construction of measures (Carathéodory's approach and Munroe's Method I and Method II), Hausdorff and packing measures, various versions of Vitali's and Besicovitch's covering theorems, density theorems, and finally net measures. Chapter 2 ("Integrals") is a basic graduate course in integration covering product measures, integrals, convergence theorems, Radon-Nikodym's theorem, Riesz representation theorem and weak convergence. Chapter 3 ("Integrals and Fractals") is an introduction to dimensions of measures and covers potential theory and capacity dimensions, and Hausdorff and packing dimensions of measures. Chapter 4 ("Probability") gives an introduction to measure theoretical probability leading to the strong law of large numbers and the Martingale convergence theorem. Chapter 5 ("Probability and Fractals") covers various topics in fractal geometry that can be studied by probabilistic methods: dimensions of graph-directed self-similar measures, random self-similar sets, Brownian motion, and a section on multifractals.

The presentation is clear. Useful motivations and examples are presented before important definitions and details of all proofs are given. Exercises and problems are spread out through the text. Each chapter ends with a section containing further references and historical notes and hints for the exercises. The book contains a long (273 entries) and useful list of references including many recent entries (i.e. after 1990).

The book under review is more advanced than Falconer's 1990 text [20] but less ambitious and more accessible than Mattila's graduate textbook [45]. Whereas Falconer's popular textbook [20] avoids technical measure theoretical details and presents a large number of examples of fractal sets taken from many parts of mathematics, the book under review provides the reader with the proper measure theoretical foundations for the subject (but at a level that is more accessible to the beginning graduate student than the treatment found in Mattila's text [45] or in Rogers' classical text [60]) and concentrates on a more limited number of examples typically involving fractal measures rather than fractal sets; cf. Chapter 5. The book therefore fills the need for a careful, thorough and detailed treatment of the measure theoretical foundations of fractal geometry at a level that is accessible to beginning graduate students. The book is ideal for a first graduate course in geometric measure theory/fractal geometry stressing the measure theoretical foundations of the subject. If supplemented with [20], the students will learn both the rigorous measure theoretical foundations for the subject as well as meet a large number of interesting examples. Such a course could be followed by Falconer's recent and more advanced text [21] or Mattila's text [45] for a discussion of more sophisticated and advanced topics including, for example, tangent measures and rectifiability. Alternatively, Chapters 1 and 2 could be used by lecturers who want to illustrate a standard graduate course in measure theory by interesting fractal measures.

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