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*Special functions*, by George E. Andrews, Richard Askey, and Ranjan Roy, Encyclopedia of Mathematics and Its Applications, The University Press, Cambridge, 1999, xvi + 664 pp., \$69.95, ISBN 0-521-62321-9 (hardcover)

Somehow, I've ended up with three copies of this book. They are all being put to good use: one for the office, one for home, one for, well, collegiate assemblies or impactions on the Schuylkill expressway. (The book is packed with brief, challenging superveniences that make it a browser's delight.)

The book testifies to the ongoing preoccupations of the authors. Richard Askey has had a long-standing interest not just in special functions and related subjects but also in the history of mathematics. One of the delightful features of this book is how the sense of history, of mathematics being created and savored, informs the text. Each chapter has a historical introduction that serves to motivate what follows. George Andrews has an abiding interest in combinatorial mathematics, and one of the chapters, devoted to partition theory, clearly has its origin in his concerns. Ranjan Roy has worked extensively in differential equations, and that interest, too, has left its mark on the book. Throughout there is a wealth of references, references that affirm the catholic interests and experience of the authors.

There are topics you will not find in this book. There is almost nothing on modular functions, nor on functions satisfying differential equations with more than three singular points (Heun functions, for example), nor anything on uniform asymptotic approximations, nor much on the family of incomplete gamma functions. The book is mostly silent about the applications of special functions to the hard physical sciences— diffraction, heat conduction, electromagnetism— topics that make the book of Lebedev [leb] so rewarding. But the book has an abundance of material that has been long neglected by the authors of other works on special functions, material overdue in an exposition aimed at the general reader. Furthermore, it is virtually the only book on the subject to pay homage to a wide variety of truly contemporary results and methods. It has a very generous serving of exercises—exercises that provoke, that illuminate, that encourage one to make one's own discoveries in the subject at hand.

The book offers a cornucopia of proofs, often two or three proofs of a single result. I was charmed and surprised by these proofs, and often I found the proofs that were historically the earliest were the most imaginative and unexpected. As the authors must have perceived, proofs in the field of special function occupy a privileged place. They are more than instruments for discovering new truths; they often point the way to generalizations of known truths, and may even suggest the way to extend concepts. Sometimes one proof may generalize, while another does not.

The book possesses a unity of vision that gives it an intellectual coherence rare among its fellows and makes the authors more than merely cicerones to a zoo of the unusual. Overall, the level of writing is unusually high: measured, highly motivated, lucid. At times the authors make what seem to me to be unfortunate choices. For instance, the proleptic use of important concepts defined only later

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in "remarks" is not a good tactic. However, as with any book offering such an abundance, there will inevitably be things to cavil about. What I consider to be defects are minor compared to the nature of the achievement. So much in the field of special functions depends on that unquantifiable but crucial feature called *taste*. The material in this book reflects the impeccable taste of its authors.

Askey has remarked that a special function is simply a mathematical function that has been used often enough to deserve a name. The special functions in this book are almost exclusively functions of hypergeometric type and their immediate generalizations. A generalized hypergeometric function is a complex-valued multi-parameter creature which can be defined several ways: as a Mellin-Barnes contour integral (my favorite), as a solution of a certain higher order linear differential equation with polynomial coefficients (sort of unworkable), or, most commonly, as the Maclaurin series,

$$ {}_pF_q\left( \begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix} ;z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_p)_n}{(b_1)_n (b_2)_n ... (b_q)_n} \frac{z^n}{n!}, \tag{A} $$

where no $b_j$ is a negative integer and

$$ (a)_n = \left\{ \begin{matrix} 1, n = 0; \\ a(a+1)(a+2)...(a+n-1), n > 0 \end{matrix} \right\}. $$

The series above converges for all $z$ if $p \leq q$ and for $|z| < 1$ if $p = q + 1$. The series diverges (unless it terminates) if $p > q + 1$, although a meaning can then be assigned to it by means of a contour integral. If a numerator parameter is a negative integer, say, $-n$, the series terminates after $n + 1$ terms. The series is called hypergeometric since it is an obvious generalization of the geometric series ($q = 0, p = 1, a_1 = 1$). The emphasis is not misplaced. Hypergeometric functions are the most ubiquitous and useful mathematical functions, both practically and theoretically (they are instrumental in the proof of, for instance, the Bieberbach conjecture and the irrationality of $\varsigma(3)$). The most recent inquiry I have received about a function of the form ${}_4F_3$ with $z = 1$, one of the most salient hypergeometric functions, was penned by a physicist who had encountered the function in a study of Feynman diagrams. More recently functions not of hypergeometric type have become important, for instance, those functions satisfying a Korteweg-deVries equation, commonly called solitons, [dra], or Heun functions, which occur as eigenfunctions of certain Schrödinger operators. Often, though, these functions can be related to functions of hypergeometric type; for instance, certain solutions of one Korteweg-deVries equation are elliptic functions which are, in turn, inverses of hypergeometric functions, and the coefficients of the Maclaurin series for Heun functions are orthogonal polynomials in a parameter, and such polynomials are commonly functions of hypergeometric type. Thus the book has applications far beyond its apparent territory.

I wish to examine the book in depth and point out some of its unusual features, some of its many assets, some of its infrequent faults.

**Chapter 1: The Gamma and Beta Functions.** The Gamma function, defined for $\operatorname{Re}(z) > 0$

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, $$

is arguably the most basic mathematical function, and all special function books put this function somewhere near the beginning. The Beta function, $B(x,y)$, defined for $\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$, by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$$

is similarly important and can be shown to be expressible in terms of the Gamma function by means of the formula

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{B}$$

The proof offered was not my favorite— the one found in Lebedev and based on a change of variable in a double integral. However, browsing through the copious exercises I found, "Use the change of variables $s = ut$ to show that

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-(s+t)} dt ds$$

is $\Gamma(x+y) B(x,y)$." The exercise attributes this proof to Poisson. The authors give *four* proofs of the reflection formula for the Gamma function. They give a beautiful proof of the equally beautiful Bohr-Mollerup theorem: *the Gamma function is the only positive logarithmically convex function $f$ defined on the positive reals with $f(1) = 1$ which satisfies the recurrence relation $f(x+1) = xf(x)$*. Why most special functions books fail to include this spectacular result— the proof is really brief— baffles me. For my special functions course I drew the proof from Conway's complex analysis book.

The authors give some intriguing group-theoretic generalizations of the Gamma function, but unfortunately here the exposition falters. Few in the audience towards which the book is directed will understand what is going on. Briefly, the authors talk about additive and multiplicative characters of the groups of integers mod $p$, $p$ a prime, denoted, respectively, by $\chi$ and $\psi$. The reader, confused by the talk of isomorphisms and the use of undefined group-theoretic symbols, will probably assume that the characters for the multiplicative group can be recovered as effortlessly as those for the additive group, when the construction even for small $p$ can be tricky; see [apo]. It is untimely to hit the reader with the notation *id* before the identity (principal) character is defined. The authors first should have defined a multiplicative character, then given an example, say, the Legendre symbol $\left(\frac{n}{p}\right)$, or the entries in one of the tables in Apostol [apo]. They could then have defined the Gauss sum as follows:

$$\sum_{n=0}^{p-1} \chi_i(n) \, e^{2\pi jn/p},$$

which is what other books do, [red], and just observed that this provides an analogue of the Gamma function since, in the integral for the Gamma function, the quantities $t^x$ and $e^{-t}$ are really group characters, i.e., homomorphisms respectively from the multiplicative group $\mathrm{R}^+$ and the additive group R to the complex numbers.

I don't much care for the demonstration that the two functions $e^{-ct}$ and $t^c$ constitute the only continuous solutions of the functional equations

$$f(x+y) = f(x)f(y), \qquad f(xy) = f(x)f(y),$$

respectively. These statements can be deduced in a straightforward way from the solutions of Cauchy's equation,

$$f(x+y) = f(x) + f(y), \tag{C}$$

and those solutions can be found assuming only local integrability of $f$. Integrate (C) and use the properties of the integral to get

$$yf(x) = \int_0^{x+y} f(t)\,dt - \int_0^{x} f(t)\,dt - \int_0^{y} f(t)\,dt.$$

The right hand side is symmetric in $x$ and $y$, so interchanging them gives

$$xf(x) = xf(y) \quad \text{or} \quad \frac{f(x)}{x} = \frac{f(y)}{y} = const., \quad \text{or} \quad f(x) = cx.$$

This sly derivation is due to Shapiro [sha]. It is known that (C) has discontinuous solutions also; the construction of these solutions requires the use of a Hamel basis for the reals. Any discontinuous solution is truly bizarre: it is unbounded in any interval, and its graph is dense in $R^2$.

There has for some time been a growing interest in the use of probabilistic methods for deriving results for special functions. The crucial tool is the central limit theorem; see [goh], for example. Van Assche's book [van] provides other examples, including an astonishing application of the theory of the distribution of sums of independent random variables to the derivation of the asymptotic properties of the Jacobi polynomials. In this contemporary mode, the authors give a very enjoyable probabilistic derivation of the expression (B) for the Beta function.

Tucked away in the splendid bounty of 56 exercises for this chapter, I found the following suggestion for a generalization of the Bernoulli polynomials. As above, let $\chi$ be a multiplicative character for the reduced residue classes mod $p$, and write (I generalize a bit)

$$\frac{te^{xt-t}\sum_{n=1}^{p-1}\chi(n)\,e^{nt}}{e^{(p-1)t}-1} = \sum_{m=0}^{\infty} B_{n,\chi}(x)\,\frac{t^n}{n!}.$$

The $B_{n,\chi}(x)$ are generalizations of the Bernoulli polynomials and reduce to the latter when $\chi$ is the principal character, $\chi \equiv 1$. When one reflects on the many uses the Bernoulli polynomials are put to in analysis, the mind boggles. Does the above lead to useful extensions of the Euler-Maclaurin summation formula? Apparently not, but manifold other questions arise. Where are the zeros of these polynomials, for instance, and how do they depend on $\chi$?

**Chapter 2. Hypergeometric Functions.** In this chapter the authors first demonstrate that many important elementary functions may be obtained as special cases of the general hypergeometric function (A). A great deal of attention is devoted to the function ${}_2F_1$, called Gauss' hypergeometric function, and its transformation theory is elaborated by the use of Riemann's brilliant concept of a $P$ function. Few other books do this. The authors give most of the traditional properties and relations for this function. There is an excellent section on the dilogarithm function,

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x\,{}_3F_2\left( \begin{matrix} 1,1,1 \\ 2,2 \end{matrix} ;x \right), \qquad |x| < 1.$$

This section contains a bestiary of strange results, for instance,

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \left(\ln\left(\frac{\sqrt{5}-1}{2}\right)\right)^2.$$

Other topics treated are binomial sums, Dougall's bilateral sum, i.e.,

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(d+n)},$$

which can be evaluated in terms of Gamma functions, and fractional integration. This chapter has 44 meaty exercises.

**Chapter 3. Hypergeometric Transformations and Identities.** This chapter talks about quadratic transformations of Gauss' function and that function's connection with elliptic integrals and the famous algorithm of the arithmetic-geometric mean, defined by $a_0 = a > 0, b_0 = b > 0$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, 2\dots.$$

This algorithm converges with startling rapidity (quadratically), and $a_n$, $b_n$ have a common limit, call it $M(a,b)$. Recently there have appeared in the literature some very dramatic algorithms for computing the transcendental number $\pi$; see [bor]. The present authors give an example of one:

$$\pi = \frac{M^2\left(\sqrt{2},1\right)}{1 - \sum\limits_{n=0}^{\infty} 2^n c_n^2}$$

where $c_n^2 = a_n^2 - b_n^2, a_0 = 1, b_0 = 1/\sqrt{2}$.

The authors next present some results for higher order hypergeometric series with argument $z = 1$, arcane results previously available only in specialized references, such as Bailey [bai], or Slater [sla], or the Bateman volumes [erd]. These results, eponymous for neglected British mathematicians such as Whipple, Dixon and Dougall, deserve to be far better known than they are. Like many other results developed at the turn of the century and afterwards consigned to the quaint or negligible, they are enjoying an appreciable currency nowadays in the world of physics.

One of the most dramatic sections in this chapter is devoted to indefinite hypergeometric summation, a subject treated in no other book on special functions. The discoveries in this area have, literally, transformed many branches of mathematics, and computations formerly intractable and results formerly undreamt of have, through their applications, become almost routine.

A sequence $s_n$ is said to be *hypergeometric* if its term ratio $s_{n+1}/\,s_n$ is a rational function of $n$. One of the first results is an algorithm due to Gosper, which seeks to determine whether a finite sum of the form $S_n = \sum_{k=0}^{n} s_k$ is itself expressible as a hypergeometric sequence. To the uninitiated reader, this may seem to be a niggling concern; that hypothetical reader will just have to accept my assurance that the issue is of monumental importance in many areas of mathematics and science.

Later, Doron Zeilberger and Herbert Wilf developed a set of algorithms that generalize the Gosper method, algorithms in which the sequence $s_n = s_{n,k}$ depends on both the variables $n$ and $k$ and is hypergeometric in both. The algorithm code named {zeil} finds a linear difference equation for $S_n$ whose coefficients are polynomials in $n$. The algorithm {hyper} determines whether this recurrence has solutions which are hypergeometric in nature. Both algorithms are explained in the idiosyncratic and fascinating book $A = B$ [pet]. Their implementation as MAPLE or MATHEMATICA programs may be downloaded from the creators' websites. I have used these algorithms profusely, most recently to obtain a simple explicit construction for the so-called associated Legendre polynomials, which are useful in numerical quadrature and Padé approximation.

**Chapter 4. Bessel Functions and Confluent Hypergeometric Functions.** This chapter, by and large, treats material that is available in many other sources, but it has unusual features that really held my interest, for instance, the detailed examination of the zeros and the monotonicity properties of the functions. One very welcome result is a theorem due to Saff and Varga concerning zero-free regions of polynomials satisfying 3-term recurrence relations. Historically, one of the first applications of this theorem was to describe the zero-free regions of the partial sums of the Maclaurin series for $e^z$.

**Chapter 5. Orthogonal Polynomials.** Let $\phi(x)$ be a distribution, i.e., a non-decreasing function with an infinite number of points of increase all of whose moments, which are the integrals $(1,\ t^n)$ (see below) exist. We may define a set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ , where $p_n(x)$ is a polynomial in $x$ of exact degree $n$, by requiring

$$\int_a^b p_m(x)\, p_n(x)\, d\phi(x) = h_n \delta_{m,n}, \qquad h_n \neq 0, \qquad m, n = 0, 1, 2, \ldots .$$

This set of polynomials is said to be orthogonal (with respect to the above distribution). One of the most general polynomials is a terminating series of hypergeometric type, the Jacobi polynomial:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} ; \frac{1-x}{2} \right), \qquad \alpha > -1, \quad \beta > -1.$$

For these polynomials, $d\phi(x) = (1-x)^{\alpha}(1+x)^{\beta}\, dx$ and $[a, b] = [-1, 1]$.

This chapter treats general orthogonal polynomials and topics such as Gaussian quadrature, zeroes of the polynomials, recurrence relations, continued fractions. A useful result concerns the moment generating function

$$H(x) = \sum\nolimits_{n=0}^{\infty} (1, t^n)\, x^n$$

where

$$(1, t^n) = \int_a^b t^n d\phi(t) .$$

The authors show how a continued fraction representation of $H(x)$ may be constructed from the coefficients in the recurrence relation for the corresponding set of orthogonal polynomials.

**Chapter 6. Special Orthogonal Polynomials.** For example, Laguerre and Hermite polynomials. We have here a description of Jacobi polynomials based on the use of Gram determinants, i.e., determinants of the form $|c_{i+j}|_{i,j=0..n}$, where $c_n = (1, t^n)$. I have found Gram determinants to be an invaluable and under-appreciated tool for treating not only polynomials orthogonal with respect to a distribution but polynomials orthogonal with respect to more complicated inner products, such as Sobolev polynomials. I am happy to see their abundant deployment here. A tasty morsel is the evaluation of the special determinant

$$\Delta_k = |1/(a+i)_j|_{\substack{0 \le i \le n-1 \\ 0 \le j \le n, j \ne k}} .$$

There is a discussion of completeness of sets of orthogonal polynomials. The authors derive the asymptotic behavior of the Jacobi polynomials for large $n$ by the use of Darboux's method. This method is crucial in asymptotic analysis, and I dislike the fact that the authors bury the method in text rather than stating it explicitly as a theorem. There is a detailed study of the problem of linearization of a set of orthogonal polynomials, i.e., finding the coefficients $a(k, m, n)$ in the expansion

$$p_m(x)\, p_n(x) = \sum_{k=0}^{m+n} a(k, m, n)\, p_k(x) .$$

An explicit representation of the coefficients can be obtained for certain classes of polynomials, for instance, the Gegenbauer polynomials.

One section deals with some applications of orthogonal polynomials to problems in combinatorics. Combinatorial methods have proved unusually effective in deducing properties of special functions, and this chapter gives the reader an introduction to their use by establishing a combinatorial interpretation of the integral

$$\int_{-\infty}^{\infty} \prod_{r=1}^{k} H_{n_r}(x) e^{-x^2} dx,$$

which produces its evaluation. (The $H_{n_r}(x)$ are Hermite polynomials.)

In the field of orthogonal polynomials there is a rising hierarchy of polynomial families, each being a generalization of the ones below it. For instance, the Chebyshev polynomials are specializations of the Gegenbauer polynomials, which are specializations of the Jacobi polynomials. How far into the empyrean can one go? The most comprehensive system to date is described by the array called the *Askey-Wilson tableau* (now even available as a wall poster) presided over by the most general polynomials of hypergeometric type: the Wilson polynomials. These polynomials are a four parameter family of polynomials orthogonal on $[0, \infty)$ with respect to a very sophisticated distribution

$$d\phi(x) = \left| \frac{\Gamma(a+ix)\,\Gamma(b+ix)\,\Gamma(c+ix)\,\Gamma(d+ix)}{\Gamma(2ix)} \right| dx.$$

By taking appropriate limits, one can obtain from the Wilson polynomials dozens of important classes of orthogonal polynomials, including the Jacobi polynomials as well as many whose distributions have discrete support (points of increase) that are of great value in statistics and numerical analysis. The construction of the Wilson polynomials was a major achievement in the field of orthogonal polynomials, and their treatment is a unique feature of this book.

**Chapter 7. Topics in Orthogonal Polynomials.** The connection coefficient problem is to express a set of basis polynomials as a linear combination of polynomials from some other basis. When the polynomials are orthogonal, especially if they are of hypergeometric type, this can often be done. For instance, the coefficients in the expansion

$$P_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} c_{n,k} P_k^{(\alpha,\beta)}(x)$$

can be found explicitly (in terms of hypergeometric functions of the form ${}_3F_2$ with argument $z = 1$). It is important to know when these coefficients are positive, and this leads the authors to consider a plethora of positivity results. The subject of positivity is one of great current activity. Fejér conjectured that

$$\sum_{k=0}^{n} \frac{\sin(k+1)\theta}{k+1} > 0, \qquad 0 < \theta < \pi,$$

a fact subsequently established by Jackson in 1911. The above equation can be written

$$\sum_{k=0}^{n} \frac{P_k^{(1/2,1/2)}(\cos\theta)}{P_k^{(1/2,1/2)}(1)} > 0, \qquad 0 < \theta < \pi,$$

and this clamors for generalization. The authors do just that, considering the positivity in a number of instances of the sum

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\cos\theta)}{P_k^{(\beta,\alpha)}(1)}.$$

(The superscripts in the denominator are *not* in error.) The positivity of this sum for $\beta = 0$ and $\alpha = 0, 1, 2, \ldots$ was required in deBranges's celebrated proof of the Bieberbach conjecture.

Another arresting result is Vietoris's inequality. Define

$$c_{2k} = c_{2k+1} = \frac{1}{2^{2k}}\binom{2k}{k}, \qquad k \geq 0.$$

Then

$$\sum_{k=1}^{n} c_k \sin kx > 0, \quad \sum_{k=1}^{n} c_k \sin kx > 0, \quad 0 < x < \pi.$$

Plotting the curves of the sums above using, say, MAPLE leads to all sorts of speculations.

A section on the irrationality of $\zeta(3)$,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

is breathtaking, one of the high points of the book. By anyone's standards, this is beautiful mathematics. Apery (in 1978) proved that $\zeta(3)$ is irrational. The authors present another simpler proof due to Beuker (1979) that uses Legendre polynomials. The arguments produce a lemma from which the assertion follows: *There exist two sequences of integers,* $\{A_n\}, \{B_n\}$, *with the property*

$$0 < |A_n + B_n \zeta(3)| < 3\left(\frac{9}{10}\right)^n.$$

Note that if $\zeta(3) = p/q$, then the sequence of positive numbers $|A_n + B_n\zeta(3)|$ must be $\geq 1/q$. But the right hand side of the above equation goes to zero as $n \to \infty$.

**Chapter 8. The Selberg Integral and Its Applications.** I am delighted to see this fascinating topic covered in a book. The Selberg integral is a generalization of the Beta integral,

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1} (1-x_i)^{\beta-1} |\Delta(x)|^{2\gamma} \, dx_1 dx_2 \ldots dx_n,$$

where

$$\Delta(x) = \prod_{1 \leq i < j \leq n}^{n} (x_i - x_j).$$

The authors give Aomoto's evaluation of a more general integral, as well as a proof due to Anderson. This integral has many applications, including one to a problem in electrostatics posed by Stieltjes. Place charges of size $p$ at 0 and of size $q$ at 1 and unit charges at $x_1, x_2, \ldots, x_n, 0 < x_j < 1$. What is the equilibrium position of the $n$ charges? It turns out that the $x_i$ will be the zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(1-2x)$. Selberg's integral can be used to obtain the equilibrium energy of the system.

A section is devoted to constant term identities, which are being much discussed nowadays. A typical problem, whose solution the authors present, is to find the constant term in the expression

$$\prod_{j \neq l} \left(1 - \frac{z_j}{z_l}\right)^{a_j},$$

where the $a_j$ are integers.

**Chapter 9. Spherical Harmonics.** Harmonic polynomials are homogeneous polynomial solutions of Laplace's equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

The functions which are restrictions of harmonic polynomials to the sphere in $R^n$ are called spherical harmonics. The authors treat these functions in depth: addition theorems, orthonormality, Fourier transforms. This chapter offers the authors an opportunity to introduce one of the most powerful methods for obtaining results in special functions, namely, group representation theory. It turns out that spherical harmonics are irreducible representations of $SU(2)$, the group of all complex matrices of the form

$$\begin{pmatrix} b & d \\ -\bar{d} & \bar{b} \end{pmatrix}$$

with determinant 1. Anyone who wants a relatively painless introduction to the subject need look no further. These sections are a model of clarity and are equaled in their effectiveness only by the little Dieudonné pamphlet published by the AMS, [die].

**Chapter 10. Introduction to $q$-Series.** $q$-Series are a generalization of hypergeometric series. In the latter, the term ratio is a rational function on the summation variable $n$. In $q$-series, the term ratio is a rational function of $q^n$ where, usually, $0 \leq q < 1$. The $q$-analog of the symbol $(a)_n$ is

$$(a;q)_n = (1-a)(1-aq)\left(1-aq^2\right)\ldots\left(1-aq^{n-1}\right).$$

The computations can be arranged so that in the limit as $q \to 1$ the term ratio becomes a rational function of $n$ and the series becomes a hypergeometric series. There is a very nice little introduction that motivates the subject using a combinatorial setting based on lattice paths in $R^2$. This study leads to a $q$-extension of the binomial theorem. The $q$-integral is a creation with an august mathematical history: Fermat (and even earlier, Archimedes, for a special case) used essentially this tool to compute $\int_0^a x^\alpha dx$. The $q$-integral is

$$\int_0^a f(x)\, d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n)\, q^n.$$

Note the limit as $q \to 1^-$ the integral becomes $\int_0^a f(x)\, dx$. For $f(x) = x^a$ the series on the right is a geometric series and so can be evaluated.

$q$-Series are of crucial importance in combinatorics and number theory and are increasingly welcome in the abstruse world of particle physics. One result of supreme utility in number theory is the triple product identity:

$$(x;q)_\infty (q/x;q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} x^k.$$

There is a $q$-Gamma function which satisfies the functional equation

$$f_q(x+1) = \frac{(1-q^x)}{(1-q)} f_q(x), \qquad f_q(1) = 1,$$

and an analog of the Bohr-Mollerup theorem, too. $q$-Hypergeometric series are called basic hypergeometric series, and their theory is quite well developed, [gas].

This chapter is unique. No other book on special functions (with the exception of specialized treatises) discusses so extensively the topic of $q$-series. Many of the results quoted are due to Ramanujan, but many other great mathematicians—Gauss, Cauchy, Jacobi— have had a hand in the development of the subject. There is a close connection with theta functions, and hence with elliptic functions, so the authors' subsequent treatment of those functions has great logical coherence. For me this chapter alone justifies the cost of the book.

**Chapter 11. Partitions.** One of the authors, George Andrews, is an authority on this subject; see [and]. The material on $q$-series segues effortlessly into partition theory, and a typical result illustrates the connection: *Let $Q_m(n)$ denote the* *number of partitions of n into exactly m distinct parts. Then*

$$\sum_{n=0}^{\infty} Q_m(n)\, q^n = \frac{q^{m(m+1)/2}}{(1-q)(1-q^2)\ldots(1-q^m)}.$$

The authors discuss graphical methods, many $q$-series identities, and the congruence properties of partitions.

**Chapter 12. Bailey chains.** There is a class of $q$-identities called Rogers-Ramanujan identities, named after the great Ramanujan and the underrated British mathematician L. J. Rogers, who published his seminal work in 1917. The Rogers-Ramanujan identities are quite pretty, but too technical to reproduce in a review. The English mathematician W. N. Bailey discovered in the 1940's a method for obtaining results on hypergeometric series, and the method can be adapted to $q$-series, and hence to the proof of the above identities. The result, Bailey's lemma, is as powerful (intelligently applied) as it is easy to prove: *Subject to convergence, if*

$$\beta_n = \sum_{r=0}^{n} \alpha_r U_{n-r} V_{n+r}, \qquad \gamma_n = \sum_{r=n}^{\infty} \delta_r U_{r-n} V_{r+n},$$

*then*

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

(Slater [sla] states that this result is implicit in the work of Abel, more than one hundred years earlier.) The pair $(\alpha_n, \beta_n)$ is called a Bailey pair if they are related as in Bailey's lemma. Given a Bailey pair, a new Bailey pair can be produced, so by successive application of the lemma, one can produce a sequence of pairs, $(\alpha_n, \beta_n) \to (\alpha'_n, \beta'_n) \to (\alpha''_n, \beta''_n) \to (\alpha'''_n, \beta'''_n) \ldots$, and thus a sequence of special function series identities. Bailey's lemma has proved to be a philosopher's stone of $q$-series. It takes great experience to apply it productively, but the results can be dramatic.

The book closes with 6 appendices: infinite products, summability and fractional integration, asymptotic expansions, the Euler-Maclaurin summation formula, the Lagrange inversion formula, and series solutions of differential equations.

The book is gorgeously composed and typeset, but there are loads of mistakes, everything from the mislabelling of equations to the assertion that the elliptic function $nc(u, k)$ is its own reciprocal. This serves to confirm my suspicions that mathematical text has a life of its own and at night, in the solitude of the publisher's drawers, morphs into the unrecognizable. (I want to assure the nervous reader, however, that there are no mistakes in this review.) Richard Askey has told me that a paperback edition of the book is in the offing. No doubt, mistakes will be expunged from future editions, and there will certainly be future editions. One measure of the success of a mathematical book is: does it give the reader ideas, ideas as lush and provocative as those one gets from a stimulating conference? By that criterion alone, this book is way over the top. This is a splendid work, and I predict that it will be a bestseller as well.

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