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Real submanifolds in complex space and their mappings, by M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999, xii + 404 pp., \$69.50, ISBN 0-691-00498-6

By the Riemann Mapping Theorem, if Ω is a bounded simply connected domain in $\mathbf{C}$, there is a one-to-one *holomorphic* or complex analytic map of Ω onto the unit disc $\{z \in \mathbf{C} : |z| < 1\}$. The map is *biholomorphic*, that is, holomorphic with a holomorphic inverse. If Ω has a $\mathcal{C}^\infty$ boundary $\partial\Omega$, the map extends to a one-to-one $\mathcal{C}^\infty$ map on $\overline{\Omega}$. If in addition $\partial\Omega$ is real analytic, the map extends to be holomorphic on a neighborhood of $\overline{\Omega}$. Thus, any two bounded simply connected domains in $\mathbf{C}$ are biholomorphically or complex analytically equivalent, and, under reasonable hypotheses on the boundaries of the domain, the equivalence behaves nicely at the boundary.

The results on boundary behavior can be localized. If M is a real analytic curve in the plane $\mathbf{C}$, there is a biholomorphic change of coordinates taking a small piece of M to an interval on the real axis. This is a simple consequence of the Riemann Mapping Theorem and the Schwarz Reflection Principle. Thus, the curve has no interesting local geometry or complex structure.

If γ is an analytic simple closed curve in $\mathbf{C}$ and f is an analytic function on γ, f does not necessarily extend to a holomorphic function on the bounded component of the complement of γ, as is shown by the example $f(z) = 1/z$ on $\{z \in \mathbf{C} : |z| = 1\}$. If $\Omega = \mathbf{C} \setminus \{0\}$, the function $f(z) = 1/z$ is holomorphic on Ω but not on any larger domain. In fact, if Ω is a domain in $\mathbf{C}$, there is a function which is holomorphic on Ω but does not extend to be holomorphic on any larger domain.

In several complex variables, the situation is very different. Poincaré observed in 1907 that there is no biholomorphic map from the bidisk $\{(z, w) \in \mathbf{C}^2 : |z|, |w| < 1\}$ to the ball $\{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 < 1\}$. Thus, the Riemann Mapping Theorem fails in $\mathbf{C}^n$ for $n \geq 2$.

Suppose M and M' are real analytic real hypersurfaces through the origin in $\mathbf{C}^n$, $n \geq 2$. Poincaré showed that in general it is impossible to find a biholomorphic change of coordinates in a neighborhood U of the origin mapping $M \cap U$ into M'; one cannot even find such a formal power series; there are an infinite number of "geometric invariants" which are preserved by such maps.

In 1906, Hartogs proved a striking extension theorem.

Theorem 1 (Hartogs' Extension Theorem). *Let Ω be an open set in $\mathbf{C}^n$, $n \geq 2$, and let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. If g is a holomorphic function on $\Omega \setminus K$, there is a holomorphic function G on Ω such that $G\big|_{\Omega\setminus K} = g$.*

Suppose M is a $\mathcal{C}^\infty$ real hypersurface in $\mathbf{C}^n$, $n \geq 2$, with defining function ρ, $M = \{z : \rho(z) = 0\}$ and $d\rho \neq 0$ on M. Let

$$T'(M,p) = \{Z = \sum a_j \frac{\partial}{\partial z_j} \in T(M,p) \otimes \mathbf{C} : Z\rho = 0\}, \tag{1}$$

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and let

$$T''(M,p) = \{\, \overline{Z} : Z \in T'(M,p)\}. \tag{2}$$

If Ω is a domain with boundary M and f is a holomorphic function on Ω, then f satisfies the Cauchy-Riemann equations in Ω, $\frac{\partial f}{\partial \bar{z}_j} = 0,\ j = 1, \dots, n$. Hence, if f extends to be $\mathcal{C}^\infty$ on $\overline{\Omega}$, then f is a *CR function*; i.e., it satisfies the *tangential Cauchy-Riemann equations* on M, $\overline{Z} f = 0$ on M for all $p \in M$ and $\overline{Z} \in T''(M,p)$.

A remarkable result of Bochner (1943) gives a converse in case the boundary is compact and connected and yields Hartogs' Extension Theorem as an immediate consequence.

Theorem 2 (Bochner's Extension Theorem)**.** *Let Ω be a bounded open set in $\mathbf{C}^n$, $n \geq 2$, with $\mathcal{C}^\infty$ boundary M and suppose that $\mathbf{C}^n \setminus \overline{\Omega}$ is connected. If $f \in \mathcal{C}^\infty(M)$ satisfies the tangential Cauchy-Riemann equations, there is a unique function $F \in \mathcal{C}^\infty(\overline{\Omega})$ such that $F\big|_M = f$ and F is holomorphic on Ω.*

Bochner's Extension Theorem is global. The first local version was proved by Lewy in 1956. Its statement requires the Levi form, the first of Poincaré's geometric invariants, introduced by E. E. Levi in 1910. The Levi form is the hermitian form $\mathcal{L} = \mathcal{L}_\rho$ on $T'(M,p)$ defined by

$$\mathcal{L}(Z,W) = \partial\overline{\partial}\rho(Z,\overline{W}).$$

Theorem 3 (Lewy's Extension Theorem)**.** *If $\mathcal{L}$ has a nonzero eigenvalue at $p \in M$, and f is a $\mathcal{C}^\infty$ function which satisfies the tangential Cauchy-Riemann equations on a neighborhood of p in M, there is a neighborhood U of p such that f extends to be holomorphic on one side of M in U.*

A domain Ω with $\mathcal{C}^\infty$ boundary is strictly pseudoconvex if there is a defining function $\rho \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\Omega = \{z : \rho(z) < 0\}$ and the Levi form $\mathcal{L}_\rho$ is positive definite on the boundary of Ω. Although there is no Riemann mapping theorem in several complex variables, Charles Fefferman (1974) proved the following result on boundary regularity of biholomorphic maps in the strictly pseudoconvex case.

Theorem 4 (C. Fefferman)**.** *Let Ω and Ω' be bounded strictly pseudoconvex domains in $\mathbf{C}^n$ with $\mathcal{C}^\infty$ boundaries, and let $\phi : \Omega \to \Omega'$ be a biholomorphic map. Then ϕ extends to a $\mathcal{C}^\infty$ map $\widetilde{\phi} : \overline{\Omega} \to \overline{\Omega'}$.*

A local version of this was proved independently by Lewy (1977) and Pinchuk (1975) in the analytic case, then by Nirenberg, Webster and Yang (1980) in the $\mathcal{C}^\infty$ case, using a reflection principle. Webster (1977) showed that if M and M' are Levi nondegenerate algebraic real hypersurfaces in $\mathbf{C}^n$, $n \geq 2$, and f is a biholomorphic map defined in a neighborhood of M with $f(M) \subset M'$, then f is algebraic.

The book under review considers important problems related to these results. The spaces $T'(M,p)$ and $T''(M,p)$ can be defined as in (1) and (2) on any submanifold M in $\mathbf{C}^n$. The submanifold M is a *CR submanifold* if the dimension of these spaces is constant, in which case this dimension is the *CR dimension* of M. CR functions on a CR submanifold are then defined as in the case of a real hypersurface. The main focus of the book is major progress in the study of CR manifolds, CR functions and CR maps in the last twenty years. The authors use a wide range of techniques, including techniques from harmonic analysis, ordinary differential equations, partial differential equations, differential geometry, Lie groups, commutative algebra, complex algebraic geometry and real algebraic geometry as well as several complex variables.

The problems considered in the book involve CR submanifolds and extensions and regularity of holomorphic functions, CR functions and CR maps. Even in the case of real hypersurfaces, it is not true in general that every CR function extends to one side as a holomorphic function. For example, the function $f(z_1, \ldots, z_n) = e^{-1/(\operatorname{Re} z_n)^2}$ is a CR function on the hypersurface $M = \{\operatorname{Im} z_n = 0\}$, but it is not the boundary value of a holomorphic function defined on one side of M in any neighborhood of 0. Thus, some sort of nondegeneracy is required for many of the results in the book. The main nondegeneracy conditions used in the book are minimality, essential finiteness, finite nondegeneracy and holomorphic nondegeneracy, all conditions which were introduced in the last fifteen years. The book contains an extensive discussion of these conditions and the relations among them.

How far can the Lewy Extension Theorem be generalized? What is a necessary and sufficient condition on a generic CR submanifold M for holomorphic extendability of all CR functions on M to an open wedge with edge M? Under what conditions do all nondegenerate CR maps from a CR manifold M to a CR manifold M' of the same dimension and CR dimension extend to holomorphic maps defined in a neighborhood of M? What is the structure of the space of all CR maps from M to M'? Suppose M is a real algebraic submanifold and f is a holomorphic map defined in a neighborhood of M which takes M to a real algebraic submanifold M' of the same dimension. Under what conditions must f be algebraic? These are some of the questions addressed in the book.

The book is very well written. The first few chapters contain mostly background material, including definitions of CR manifolds and CR functions, approximation of CR functions by holomorphic functions, real vector fields and their commutators, algebraic functions and algebraic manifolds, and the Hilbert transform. The authors do not assume a knowledge of several complex variables. An ambitious, well-trained second- or third-year graduate student could read the book. In addition to introducing each chapter and each section with a summary, the authors include frequent comments to ensure that readers know where they are heading and will not get lost in details. Each chapter concludes with extensive historical notes and references.

No book could include all mathematics related to real submanifolds without turning into a very lengthy treatise. The topics omitted from the book under review include the $\overline{\partial}$-Neumann problem, and the Moser normal form and Cartan-Chern-Tanaka connection on a Levi nondegenerate hypersurface. The article *Local geometric properties of real submanifolds in complex space* [BER] by Baouendi, Ebenfelt and Rothschild surveys the local equivalence problem for real submanifolds in $\mathbf{C}^n$. The local equivalence problem is the following. If M and M' are real submanifolds of the same dimension in $\mathbf{C}^n$ with $p_0 \in M$ and $p_0' \in M'$, when is there a biholomorphic map ϕ defined in a neighborhood U of p_0 such that $\phi(p_0) = p_0'$ and $\phi(U \cap M) \subset M'$? This problem is closely related to some of the topics in the book under review, and the article is a good complement to the book.

Several books on related topics have appeared in the last decade. The book [Bog91] by Boggess discusses basic properties of CR manifolds and the tangential

Cauchy-Riemann complex. It includes some results from the early 1980's on holomorphic extendability and local solvability. The book [CS00] by Chen and Shaw discusses recent progress on the $\overline{\partial}$-Neumann problem, the Cauchy-Riemann equations and the tangential Cauchy-Riemann equations using both L^2 methods and integral kernels. The book [D'A93] by D'Angelo emphasizes the interplay between real varieties and complex varieties. It uses algebro-geometric methods to study the finiteness of order of contact of complex varieties with a real hypersurface and its relationship to subelliptic estimates for the $\overline{\partial}$-Neumann problem. Jacobowitz's book [Jac90] is about the local equivalence problem for Levi nondegenerate hypersurfaces from the point of view of differential geometry, including Moser's normal form, Cartan's method of equivalence and the Cartan-Chern-Tanaka connection.

Real Submanifolds in Complex Space and Their Mappings is an excellent book. It makes a number of important topics in several complex variables accessible to non-experts for the first time. Until its publication, most of the recent research on the topic was available only in journal articles, and the necessary background was scattered through the literature of a number of different fields. As a result, experts will also find it valuable.

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