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Harmonic maps, loop groups, and integrable systems, by M. A. Guest, London Mathematical Society Student Texts 38, Cambridge Univ. Press, Cambridge, UK, 1997, xiii + 194 pp., \$14.95 (paperback), ISBN 0-521-58932-0

Harmonic maps $f : M \to N$ of Riemannian manifolds have been studied intensively by many researchers over a long period of time. The two long reports [8], [9] by Eells and Lemaire document impressively the state of the theory of harmonic maps up to 1987. Since then many new results have been proven and "yet another report" would probably be as long as the two previous reports. The theory is so rich, since even special cases are highly interesting. Just a few examples: if N is euclidean, then one obtains the classical harmonic maps of Riemannian manifolds and if $\dim(M) = 1$, then one obtains geodesics. If M is an orientable surface and h is an isometric immersion from M into euclidean three-space, then a result of Ruh and Vilms states that (M, h) has constant mean curvature if and only if the corresponding Gauss map, mapping M into the Riemann sphere, is harmonic. Finally, the harmonic maps from Riemann surfaces to compact Lie groups are closely related to the "non-linear sigma models" and the "chiral models" investigated intensively in the physics literature.

These examples suggest that it is already very important to study the harmonic maps from Riemann surfaces to compact Lie groups and to compact symmetric spaces. And this is the case the book under review discusses. It is fortunate that for harmonic maps starting from surfaces there are several additional features which make the theory particularly beautiful and rich. The heart of the matter is the observation that in this case the harmonic map equation can be treated like an "integrable system" [14]. This makes available a large set of techniques which were originally developed to treat completely different differential equations, like the KdV equation.

While the treatment of integrable systems involved a parameter (loop), the systematic use of infinite dimensional Lie groups (loop groups) and infinite dimensional (Grassmannian like) manifolds of solutions appeared progressively in the work of Sato (and his school) [16], [11], [12]; Segal (and his co-authors) [17], [18], [15]; and Uhlenbeck [19] during the 80's. Thus new techniques became available for the treatment of harmonic maps.

On the geometric side, and at about the same time, Wente showed that there are immersions of tori into euclidean three-space which have constant mean curvature. This counterexample to the "Hopf Conjecture" brought a lot of attention to the study of surfaces of constant mean curvature and in particular initiated much work on constant mean curvature tori, e.g. [1], [13], [10], and more generally on harmonic maps of finite type, e.g. [2], [3], [4].

Finally, it was shown that every harmonic map from a Riemann surface to a compact symmetric space can in principle be derived via "factorization of exponentials" from holomorphic data [6], [7], [5].

"With the benefit of hindsight, and a new point of view suggested in ...[7]..., the genus zero case turns out to have some aspects in common with the genus

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one case. This leads to new explicit formulae in the genus zero case, and also a way of generalizing the results of Uhlenbeck and Segal. It seems reasonable to conclude that the basis of a unifying theory has now been established. From a naive computational point of view, the basic phenomenon is that the harmonic maps considered here may be expressed in terms of 'factorization of exponentials'.

The goal of this book is "to present this unifying theory in a straightforward manner." As a matter of fact, the book under review is a very well written, easily accessible introduction to how loop group techniques are used in the description of harmonic maps from Riemann surfaces to compact Lie groups and compact symmetric spaces. It is highly suitable for a beginning graduate student or any newcomer to the field. No knowledge of Lie groups or harmonic maps is required, just some mathematical maturity as can be expected of a graduate student who is familiar with the basic concepts of linear algebra, differential equations and differentiable manifolds. While the text concentrates on what will actually be needed in applications to harmonic maps, the Bibliographical Comments at the end of some chapters point to further readings and help one to understand the historical development of the techniques in question. Finally, the reviewer found only a few mistakes, and they seem to be inconsequential.

The book consists of three parts and altogether 26 chapters and an "Epilogue". The first part is devoted to one-dimensional integrable systems. It starts with an elementary introduction to Lie groups, Lie algebras and homogeneous spaces. Chapter 4 introduces Hamiltonian systems and discusses as an example the adjoint orbits of compact Lie groups. This yields naturally an example of a Lax equation. In the following chapter general Lax equations are considered. It is shown how one can find explicitly solutions to this equation, if the group G is represented in the form $G = AB$, where A and B are subgroups with trivial intersection such that $Ad(A)$ leaves the Lie algebra of B invariant. As an application the example of Chapter 4 is revisited and the Toda lattice is introduced. This way "the" standard example of an integrable system used in this book makes its first appearance: "The term 'integrable system' is used rather loosely in the literature and a precise definition will not be given in this book....As [a] substitute for a precise definition, one could exhibit a typical example, such as the Toda lattice. Whatever an integrable system is, the Toda lattice certainly is one. For this reason, the Toda lattice plays a prominent role in this book. Another reason for choosing the Toda lattice here, however, is that it happens to be directly related to (certain kinds of) harmonic maps."

Chapter 6 rewrites the Toda lattice so that it turns into a straightforward example for the Adler-Kostant-Symes method, which is the contents of chapter 7. The concluding remarks of chapter 8 discuss further generalizations of the Toda lattice. The general discussion and in particular the explicit examples illustrate nicely the effects of choosing different group splittings and different Riemannian metrics. Moreover, the important notion of "dressing" is introduced lucidly (however, the reader needs to watch for tacit assumptions on which group acts on which if $G = AB$).

Part II, chapters 9 through 22, is "parallel to chapters 1 through 8, but uses infinite dimensional Lie groups instead of finite dimensional Lie groups, zero-curvature equations instead of Lax equations, and the two-dimensional Toda lattice (and harmonic map equation) instead of the one-dimensional Toda Lattice." The style of the text is as in Part I : explicit examples constitute a gentle introduction to otherwise very technical methods and the infinite dimensional machinery of Kac-Moody groups is treated in a very much down to earth manner. In chapter 9 harmonic maps to Lie groups are defined and in chapter 18 harmonic maps to symmetric spaces are introduced. The second half of Part II is devoted to the discussion of harmonic maps to Lie groups and symmetric spaces. Various different formulations of the harmonic map equation are given. The special case, where the domain of the harmonic map is the Riemann sphere, is discussed in detail. In particular, work of Uhlenbeck and of Segal on harmonic maps into $U(n)$ and Grassmannians is presented. In chapter 22 explicit (Weierstrass like) formulae are given by "factorizing an exponential". It should be pointed out that after the publication of this book much of the work presented at the end of Part II has been generalized to harmonic maps from the Riemann sphere to arbitrary Lie groups and arbitrary symmetric spaces [5].

Finally, in Part III harmonic maps of finite type and primitive harmonic maps (already introduced in chapter 21) are investigated. The reader should be aware of the fact that the notion of finite type as presented in this book is more general than the one usually used in the literature, where only "real" initial conditions are permitted (see the comments in chapter 24). While the proofs presented in the book all go through also for the more general notion of finite type, it is not clear what geometric meaning is associated with these harmonic maps. Thus for a first reading it may be useful to restrict to "real" initial conditions and to supplement the text of the book a bit by information from [3], [4]. Clearly, at this point the book leads into open questions. And this is what the Epilogue is about as well.

In conclusion, the reviewer considers this book to be a great addition to the literature. The book presents in a unifying way a very nice introduction to a new part of harmonic map theory, is easily accessible, fun to read and has a modest price. It is an ideal text for a beginning graduate student and any newcomer to the field.

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