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Quasiconformal Teichmüller theory, by F. P. Gardiner and N. Lakic, Mathematical Surveys and Monographs, vol. 76, Amer. Math. Soc., Providence, RI, 2000, xix + 372 pp., $89.00, ISBN 0-8218-1983-6

B. Riemann observed that a compact nonsingular curve of genus $g \geq 2$ depends on $3g-3$ complex parameters. In modern language, this means that there should exist some complex analytic space $\mathcal{R}$ of complex dimension $3g-3$ and the points of $\mathcal{R}$ should be in one-to-one (perhaps canonical) correspondence with the birational equivalence classes of such curves. The space $\mathcal{R}$, a complex orbifold, is now known as the *Riemann space*. It is the *moduli* space of compact Riemann surfaces of genus g. It has a manifold covering space: the *Teichmüller space* – the principal analytic tool to study moduli of Riemann surfaces. There is another mostly algebraic path to moduli questions followed primarily by algebraic geometers for whom Teichmüller theory is one of many tools.

The analytic approach to the problem of moduli of compact Riemann surfaces expands to include studies of infinite dimensional moduli spaces, Kleinian groups, 3 (real)-dimensional manifolds, and hyperbolic geometry. The algebraic geometric approach has had remarkable success, not only in compactifying various moduli spaces associated to compact Riemann surfaces [16],[1] but to a study of projective varieties of arbitrary (finite) dimension. There is, of course, a highly nontrivial intersection. Perhaps most dramatic is the use of Teichmüller theory by P. Griffiths [29] to uniformize algebraic varieties.

The second half of the twentieth century saw remarkable progress in Teichmüller theory, mostly as a result of the work of L.V. Ahlfors, L. Bers and their students.[2] Many of the important problems regarding the structure of the space of moduli of Riemann surfaces of finite conformal type were solved. The emphasis was on finite dimensional Teichmüller theory. In the last two decades physicists (string theorists) and complex dynamicists discovered a need for Teichmüller theory. Whereas the right Teichmüller space for physicists is still lacking, infinite dimensional Teichmüller spaces, on the back burner for years, seem to be exactly what dynamicists need. The choice of topics in the book under review are motivated in great part by an attempt to supply a set of tools that are needed in the study of iteration of interesting selfmaps of interesting topological spaces.

1. History

Teichmüller spaces existed before Teichmüller. They are already present in the work of Fricke [25][3] on families of Fuchsian groups. But Teichmüller was first to realize that the family of quasiconformal maps is the most fruitful setting for a study of moduli of Riemann surfaces. These maps are less rigid than the more obviously

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[1]Subsequently also established by analytic methods; partly as a result of work of Bers [12] and Wolpert [58].

[2]Gardiner was a student of Bers and Lakic was a student of Gardiner.

[3]The fact that the basic objects of our study are not called *Fricke spaces* is not the greatest irony in the field.

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necessary conformal maps, but flexible enough to allow the use of important analytic tools, for example, the theory of elliptic partial differential equations.

Teichmüller also introduced an important extremal problem [56] and a metric on Teichmüller space that now bear his name. The extremal problem has a uniqueness part that is established by the length-area method[4] originated by H. Grötzsch [31] and subsequently exploited to great advantage in the work of E. Reich and K. Strebel [48]. Despite the passage of time and many attempts and minor successes in special cases [13], [36], all the proofs of uniqueness rely in an essential way on Teichmüller's original argument. Existence is a different matter. Compactness properties of quasiconformal maps yield at once the existence of extremal maps. The existence question has hence been interpreted to require a characterization of such maps, usually, a description of their Beltrami coefficients involving integrable holomorphic quadratic differentials. Ahlfors [3] supplied a rigorous exposition of Teichmüller's existence argument after establishing a solid foundation for the theory of quasiconformal maps. R. Hamilton, in his 1966 doctoral dissertation and in [32], and S. Krushkal [38] supplied independent modern function theoretic proofs that also opened up new avenues for future exploration. Bers [7] showed that existence can be dispensed with after one establishes uniqueness and the manifold structure of Teichmüller space. It is an almost automatic consequence of these two facts. Teichmüller's pioneering work established a natural complete metric for Teichmüller space. He did not settle the issue of complex structure. Ahlfors [4] was the first to show that these spaces are complex manifolds. His local coordinates, at most points, are periods of abelian differentials, a point of view suggested by earlier work of H.E. Rauch [45], who computed the variation of the periods under quasiconformal deformations. Ahlfors showed that the complex structure so produced on Teichmüller space is the unique one that is compatible with the metric structure and for which the period map is holomorphic. Bers [10] embedded Teichmüller space into a Banach space of holomorphic quadratic differentials[5] using Schwarzian derivatives of univalent functions. This step introduced a seemingly different complex structure on Teichmüller spaces. It is not different, since Bers invokes Ahlfors's observations and shows [8] that the period matrix is an analytic function on Teichmüller space with this "apparently new" complex structure. What is now called the *Bers embedding* anchored the modern theory of moduli to the classical theory of univalent functions.

To more or less complete a circle of ideas, Royden [49] showed that the Teichmüller metric (in the finite dimensional case)[6] on Teichmüller space agrees with the Kobayashi metric and is thus a natural consequence of the complex structure on Teichmüller space. Royden also shows that, except for some easily understood index two extensions, a naturally defined modular (also called mapping class) group is the full group of complex analytic automorphisms of Teichmüller space.

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[4]A careful comparison of the extremal length of a family of curves in a domain to the area of that domain.

[5]Morally different from the quadratic differentials appearing in the solution to Teichmüller's extremal problem. Holomorphic and meromorphic quadratic differentials almost have nine lives in Teichmüller theory. Integrable holomorphic quadratic differentials appear in the study of extremal maps, the tangent space to Teichmüller space, foliations, and flat metrics on Riemann surfaces; bounded holomorphic quadratic differentials, in the complex structure on Teichmüller space (hence also in the tangent space) and harmonic maps.

[6]The entire discussion of this section is limited to this situation.

The naturality of the entire development is emphasized by the universal mapping properties satisfied by the Teichmüller curve [30], [17], [22], a complex fiber space over Teichmüller space with marked Riemann surfaces as fibers. The Riemann curve is, on the other hand, not universal. This brings us to the book under review.

2. Highlights

The book has at least two aims: to establish in the infinite dimensional case, using a minimal number of general principles and results, as much of the finite dimensional theory as possible and to expose the tools from Teichmüller theory needed by complex dynamics. The authors are in a very good position to achieve these goals, since they and C.J. Earle are responsible for much of the recent progress in infinite dimensional Teichmüller theory, and because of their interest in the study of iteration of some naturally occurring maps. Much of the theory described in the book involves infinite dimensional Teichmüller spaces as well as the asymptotic geometric properties of the boundary of Riemann surfaces. The latter topic requires a type of Teichmüller space that has not been studied classically and has not appeared in any previous book on the subject. Among the main topics treated in the book are: the complex structure and natural metrics, in both global and infinitesimal form, on various Teichmüller spaces; the group of holomorphic automorphisms of Teichmüller space; extremal and nearly extremal quasiconformal maps; and boundary dilatations of quasiconformal maps. Several important applications, for example, welding and earthquakes, are discussed.

Many problems in Teichmüller theory involve finding good extensions of selfmaps of the unit circle (equivalently, the extended real line) to the unit disc (upper half plane). The problem is, in most cases, equivalent to selecting a canonical representative from an equivalence class of mappings. Many tools are needed to study such issues including Teichmüller existence and uniqueness, Beurling-Ahlfors extensions, harmonic maps, Douady-Earle extensions, earthquakes. All of these topics are discussed thoroughly in this book except for the Beurling-Ahlfors extension and harmonic maps. These last two topics can be found in [6] and [44], respectively.

3. Contents

The book brings to the literature the current state of the analytic theory of Teichmüller spaces. It is a thorough report on the latest developments with a solid exposition of most of the classical foundations, many times without proof.

The first chapter, "Quasiconformal mapping", is a summary without proofs of some of the main theorems on quasiconformality and related concepts necessary for Teichmüller theory. Included are reviews of the various definitions of quasiconformal maps, a discussion of the Beltrami equation, holomorphic motions, and (in the exercises) the Ahlfors-Bers theory of the holomorphic dependence on parameters of normalized homeomorphic solutions to the Beltrami equation. The emphasis of the chapter is the idea of distortion of shape and size under quasiconformal maps and how to measure the latter.

The second chapter, "Riemann surfaces", reviews the uniformization theorem (a high point of 19th-century mathematics, whose proof was not completed until about the second decade of the twentieth century) and Fuchsian groups. The quasi-invariance of the translation length of a hyperbolic Möbius transformation under quasiconformal maps is highlighted and extremal length is studied. This chapter contains the definition of Teichmüller spaces, their Teichmüller metrics (defined using quasiconformal mappings), and the briefest introduction to the Banach space $A(R)$ of integrable holomorphic quadratic differentials (only a lemma without proof giving its dimension) on the Riemann surface R.

Chapter 3, "Quadratic differentials, part I", is a geometric analysis of the Banach space $A(R)$, its dual and predual. The analysis relies on Poincaré series, the infinitesimal Teichmüller norm, and approximations of elements of $A(R)$ by rational functions. The latter is an extension of a theorem of Bers [9], whose proof is based on a refinement of a delicate "mollifier" introduced by Ahlfors [5]. A key result of the chapter is an extension theorem for vector fields defined on closed subsets of the sphere. It is based on a theorem on equivalence of norms established in [19]. This chapter develops the theme that the tangent space to $T(R)$, the Teichmüller space of R, is identifiable with normalized vector fields defined on the limit set of a dynamical system corresponding to R satisfying a boundedness condition, forming a space naturally isomorphic to the dual space of $A(R)$.

Chapter 4, "Quadratic differentials, part II", is a study of the singular Euclidean structure induced on the surface R by a quadratic differential in $A(R)$. It is an exposition with many simplified proofs of the Reich-Strebel theory [47], [48] on quadratic differentials and extremal quasiconformal mappings based on length-area arguments. Teichmüller's Uniqueness and Existence Theorems are established, the latter as a consequence of Strebel's Frame Mapping Theorem, which gives good sufficient conditions for extremality of quasiconformal mappings. It is useful at this point to roughly state[7] Teichmüller's extremal problems: Among all quasiconformal maps $f : R_1 \to R_2$ between Riemann surfaces, find the one that is homotopic to a fixed $f_o : R_1 \to R_2$ and is closest to conformal.

The Teichmüller space of the Riemann surface R is the quotient space of an equivalence relation on the set of quasiconformal maps f of R onto other Riemann surfaces. Two such maps f_1 and f_2 are *Teichmüller equivalent* provided $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of $f_1(R)$ onto $f_2(R)$ modulo the ideal boundary of $f_1(R)$. Each Teichmüller space has a distinguished *base* point or *origin*, the equivalence class of the identity map. Use of the base point permits a normalization of many problems by reducing the study to special cases; for example, in the study of arbitrary maps between Teichmüller spaces, it suffices, in most cases, to consider only base point preserving maps. The Teichmüller space defined above is the basic object of study in the book. It and the various notions of equivalence of quasiconformal maps are thoroughly analyzed in Chapter 5, "Teichmüller equivalence", which contains a proof of the remarkable Douady-Earle Extension Theorem for quasisymmetric homeomorphism of the unit circle S^1. The Douady-Earle result provides a conformally natural map that extends a group equivariant quasisymmetric automorphism of the unit circle to a group equivariant quasiconformal automorphism of the closed unit disc $\Delta \cup S^1$. It has important applications to Teichmüller theory, giving new previously conjectured results in the infinite dimensional case and simplifying the proofs of many theorems in the finite dimensional setting, including the proof of the equivalence, under appropriate hypothesis, of homotopies and isotopies between quasiconformal maps.

Chapter 6, "The Bers embedding", is a study of the analyticity of Teichmüller spaces. A discussion of cross ratios and the Schwarzian derivative leads to the study

[7] Ignoring technical conditions to be satisfied at the ideal boundary.

of the Bers map from Teichmüller spaces to Banach spaces of bounded holomorphic quadratic differentials. The local Teichmüller theory leads to the necessity of the Krushkal-Hamilton [38], [32] condition for extremality.

Every complex analytic manifold X carries a naturally defined hyperbolic pseudometric introduced by Kobayashi. It is the largest pseudometric σ with the property that all holomorphic maps from the disc with its Poincaré metric into (X, σ) are distance nonincreasing. In Chapter 7, "Kobayashi's metric on Teichmüller space", the authors follow the approach of [21] using Slodkowski's Extension Theorem [51] for holomorphic motions to establish the equality of the Kobayashi and Teichmüller metrics on arbitrary Teichmüller spaces. This important theorem was first proven by different methods by Royden [49] for Teichmüller spaces of closed surfaces and generalized by the senior author [26] to the infinite dimensional case. This leads naturally to a study of geodesics in Teichmüller spaces.

Quasiconformal maps between Riemann surfaces induce natural *geometric* maps between Teichmüller spaces (yielding the modular group when the domain and range coincide). Chapter 8, "Isomorphisms and automorphisms", is devoted to the question whether there can be any other complex analytic maps between Teichmüller spaces. For the case of closed surfaces, Royden [49] showed that the answer is essentially no. This result is extended considerably. The authors rely on an analysis of the isometries of cotangent spaces induced by analytic automorphisms of Teichmüller spaces. The cotangent space to Teichmüller space is the double dual of the Banach space of integrable holomorphic quadratic differentials discussed in Chapter 3.

Chapter 9, "Teichmüller uniqueness", is based on [15], joint work of the junior author with three colleagues, unfortunately omitted from the bibliography. The chapter begins with a discussion of critical examples, mostly the work of K. Strebel [52], [53] (see also [46]), concerning unique extremality of quasiconformal maps. It then proceeds to its main topic, the equivalence of extremality and local extremality (involving, of course, induced maps on the contangent spaces) of quasiconformal maps. The chapter ends with a brief discussion of the nonuniqueness in Slodkowski's Extension Theorem on holomorphic motions.

The natural map from the mapping class group of a surface R to the automorphism group of the Teichmüller space of R is, in general, injective. The surjectivity of this map is discussed in Chapter 8. In Chapter 10, "The mapping class group", it is shown that the mapping class group of the surface R acts properly discontinuously on the Teichmüller space of R whenever the latter space is finite dimensional. The approach is to study the action of quasiconformal maps on the length spectrum of a hyperbolic Riemann surface (these are the logarithms of multipliers of the hyperbolic elements in the covering group of the surface). Unlike the rest of the book, this chapter's discussion applies only to finite dimensional Teichmüller spaces.

Chapter 11, "Jenkins-Strebel differentials", is an exposition of the work of [34], [54] and [55] on the Euclidean structure induced by a holomorphic quadratic differential on a Riemann surface. An associated extremal problem leads to decompositions of Riemann surfaces into ring domains that have proven to be useful in the study of extremal quasiconformal mappings and curvature properties of the Teichmüller metric, [41].

On a Riemann surface of finite conformal type, the integrable holomorphic quadratic differentials are in one-to-one canonical correspondence with equivalence classes of measured foliations. In Chapter 12, "Measured foliations", a Dirichlet type problem (minimizing a norm) is solved for a class of integrable continuous quadratic differentials. As with all Dirichlet problems, there is a uniqueness and an existence part. The uniqueness proof is based on the length-area method. The existence proof uses the Jenkins-Strebel theory of the last chapter and a theorem of W. Thurston, not proven in the book, on the density of certain classes of measured laminations. A study of the variation on Teichmüller space of a norm on foliations and an application to transverse foliations conclude the chapter.

Chapter 13, "Obstacle problems", is an application of Teichmüller theory to a mini-max problem, a transportation problem involving minimization of a Dirichlet integral constructed from quadratic differentials. It is based on the work of the senior author and R. Fehlmann [24] and gives a far-reaching generalization of the classical slit mapping theorems for planar domains.

Chapter 14, "Asymptotic Teichmüller space", is mostly based on the productive collaboration of the authors with Earle [20] and an unavailable sequel to this paper. As indicated earlier, the Teichmüller space $T(R)$ is a quotient space of an equivalence relation (described in our discussion of Chapter 5) on the set of quasiconformal maps from R to other Riemann surfaces. A map $f : R \to R_1$ is *asymptotically conformal* if it has small dilatation off compact sets of R. If one substitutes "asymptotically conformal" for "conformal" in the definition of Teichmüller space, one obtains the *asymptotic Teichmüller space* $AT(R)$. This is a nontrivial space only when the surface R is not of finite conformal type. The two basic objects for study in this setting are the natural projection $\pi : T(R) \to AT(R)$ and the fiber over the origin of $AT(R)$, $T_0(R) = \pi^{-1}(0)$. It is shown that both manifolds $T_0(R)$ and $AT(R)$ satisfy many of the same properties as the more classical $T(R)$.

Chapter 15, "Asymptotically extremal maps", is based on the work of the junior author that is only available as a preprint. In this chapter it is shown that, analogously to Teichmüller's theorem, every asymptotic equivalence class in $AT(R)$ contains an asymptotically extremal representative. This involves minimizing over equivalence classes of quasiconformal maps f the infimum over compact subsets of $E \subset R$ of the L^∞-norm of the Beltrami coefficients μ of f restricted to $R - E$: this is $H(f)$, the boundary dilatation of f; its anologue in classical Teichmüller theory is the dilatation $K(f)$ of a quasiconformal map f. In analogy with the Teichmüller extremal problem for quasiconformal maps, there is a similar problem for asymptotically conformal maps that is solved in this chapter.

In Chapter 16, "Universal Teichmüller space", the general theory is applied to the Teichmüller space of the unit disc Δ. A number of issues particular to Δ are discussed. These include the space QS of quasisymmetric homeomorphisms of the unit circle S^1 (this is the proper concept of quasiconformality in real dimension one), the closed topological subgroup $S \subset QS$ consisting of symmetric maps, welding maps and quasicircles, and symmetric quasicircles. Much of the chapter is of interest to string theorists; most of it is based on joint work of the senior author and D. Sullivan [28]. A brief discussion of the Beurling-Ahlfors [14] Extension (less natural but established earlier than the Douady-Earle Extension, discussed in Chapter 5, in the sense that it is equivariant only with respect to affine maps in contrast to the latter which is equivariant with respect to all conformal selfmaps and with different applications and surprising consequences) of quasisymmetric selfmaps of $\mathbf{R} \cup \{\infty\}$ to quasiconformal selfmaps of $\mathbb{H}^2$ is included.

Let f be a quasiconformal map defined on a plane domain Ω and let p be a point on the boundary of Ω. Chapter 17, "Substantial boundary points", contains the definition of the local boundary dilatation $H_p(f)$. A generalization to arbitrary Ω of a result of Fehlmann [23] for the unit disc that there exist p for which $H(f) = H_p(f)$ is established. The result is independent of the particular choice of embedding of Ω into $\mathbf{C}$.

Earthquakes provide another way of extending a selfmap of the unit circle to one of the closed disc. In Chapter 18, "Earthquake mappings", a finite form of Thurston's Earthquake Theorem is established for cyclic order preserving maps from finite subsets of S^1 into S^1 as well as an infinitesimal version for vector fields defined at a finite number. These results yield Thurston's general theorem[8] by limiting process and an algorithm for computing invariants of laminations, also, an extension of the infinitesimal version to Zygmund bounded vector fields defined on all of S^1. Earthquake curves were used by S. Kerckhoff [35] to solve the Hurwitz-Nielsen realization conjecture.

4. Not in the book

The choice of topics covered by the book has been dictated by the authors' tastes and expertise. It is quite reasonable for their emphasis to be different from mine. Their study is restricted to the 2-dimensional approach – hence forced to neglect the important connections to hyperbolic 3-manifolds. There is no discussion of boundaries of Teichmüller spaces (and thus no mention of compactifications of moduli space) and of C. McMullen's [42] important work on the norm of the Poincaré series operator. Even though no book can cover everything, I was particularly disappointed to find no mention of Bers's Extremal Problem: Fix a topological selfmap f_o of a topological surface S_o. What is the infimum of the dilatations of quasiconformal selfmaps f of Riemann surfaces S, where f is homotopic[9] to f_o and S is homeomorphic to S_o? When is the infimum a minimum? This natural problem was formulated by Bers in 1978. In [11], it was used to re-prove and refine Thurston's classification of surface automorphisms. The problem could have been formulated, but perhaps not solved, thirty-five years earlier. It fits naturally with the other extremal problems appearing in the book. The authors, probably omitted this topic because, unlike almost all the topics discussed, it is currently not applicable to the infinite dimensional situation. It is not too optimistic to expect an infinite dimensional version of Bers's result.

5. Some problems

It is unfortunate that this important text on a subject of interest to a large segment of the community that should have been "the" up-to-date research tool and an indispensable guide for those working in the field is marred by the presence of inaccuracies, missing bibliographical data, and a few hasty arguments – in addition to the usual expected quota of typos and small errors.

It is a matter of taste to decide what topics to cover and what prerequisites to include in a book. I prefer that a book be either self contained, after specifying a set of assumed knowledge, or refer with adequate bibliographical data to other

[8] A finite dimensional version of the theorem appears in [35]; a universal Teichmüller space version was published in [57].

[9] Again, ignoring conditions to be imposed at the ideal boundary.

books but not to articles in journals. A uniqueness theorem of Cartan is stated in Chapter 8; there is no indication in what books it appears probably because the authors do not rely on it. I suspect that it is easy to provide references to books where Cartan's theorem is proven; it may not be so easy to do so for the Earthquake Theorem of Thurston that is the basis for the last chapter and for the Thurston's density result needed in Chapter 12, because as with many results of Thurston a proof was either never published or published in papers authored by others and probably not appearing in any book.

There are mistakes that cannot be explained away as a matter of taste. Perhaps most annoying is the inadvertent omission of [15] from the bibliography. This important paper is the basis of about half of Chapter 9; the omission is compounded by the fact that significant portions of that manuscript are reproduced without even the correction of numbering of lemmas. A second source of irritation is the remark in Chapter 2 following Agard's formula for the metric on the thrice punctured sphere that asserts that the formula is equivalent to Slodkowski's Extension Theorem for four points on the sphere. They proceed to outline a proof that Slodkowski's result implies the formula for the metric, but ignore the nontrivial, at least to the reviewer, reverse implication.

Problems of the type described in the last paragraph should have been eliminated during a careful editing of the manuscript that eventually became this book, a manuscript that showed great potential.

I have been informed by the authors that they are aware of some of the errors and missing bibliographical data in the book. In fact, the omission of [15] from the bibliography was first brought to my attention by Gardiner. They have created a web page with addenda and corrections. Its address is *http://comet.lehman.cuny.edu/lakic*.

6. Other books

To the best of my knowledge, one set of lecture notes, [6],[10] was published on the subject before 1970; two books appeared in the seventies, [40] and [39]; three more in the eighties, [1], [27] and [43]; and two in the last decade of the twentieth century, [33], [50]. Three of these were considered worthy of a *Bulletin* review: [18], [37] and [2]. A book closely related to and containing many of the prerequisites for a study of many topics in Teichmüller theory [55] appeared in the eighties.

Was there a need for another book? Definitely, yes. The material in this monograph is significantly different from what has previously appeared. Of the three books that have been reviewed in this journal, Lehto's book is scholarly but limited in its scope as implied by the title, Nag's is an encyclopedic treatment of the complex analytic aspects of Teichmüller theory, and Gardiner's has been superseded by recent developments. The vitality of the subject is reflected in the fact that even after this book, there is a need for one more.

7. Concluding remarks

I am pleased this book was written; it is a real service to our community. The authors tackled a formidable issue and have produced, perhaps a little too hastily, an important addition to the literature. They accomplished quite a lot. Most

[10]A very influential book. Many of us learned the subject from the lectures of Ahlfors and Bers and this little (in size only) book.

of the topics are discussed very thoroughly; some shed a lot of new light on the material. But, a student entering the field and reading this book will certainly need to have easy access to at least one of the more classical texts, and even the experts in Teichmüller theory or related fields may want to consult the authors' web site before embarking on a serious study of the text. To realize the great promise that greets the reader in the introduction, one has to work hard and concentrate on the mathematical contents and ignore the distractions. The effort is quite worthwhile.

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