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From Brownian motion to Schrodinger's equation, by Kai Lai Chung and Zhongxin Zhao, Springer, New York, 1995, xii + 287 pp., \$109.00, ISBN 0-387-57030-6

Two developments are brought to mind by the book under review. The first is the so-called Feynman-Kac formula. The second is conditional Brownian motion. The Feynamn-Kac formula arises naturally when considering the problem of calculating the distribution of a functional of a sequence of partial sums of a sequence of random variables. For example, take $X_n, n \geq 1$ to be independent, identically distributed random variables with mean zero and variance one. Define $S_n = \sum_1^n X_j$ and consider the problem of determining the distribution of the amount of time this sequence of partial sums spends above 0. Namely, set $V(x) = 1_{(0,\infty)}(x)$ and look for the distribution of $\frac{1}{n}\sum_1^n V(S_j)$ which gives the proportion of the first n sums which are positive. Inserting a scaling factor of $\frac{1}{\sqrt{n}}$ and summing to nt rather than to n, one gets $\frac{1}{n}\sum_1^{nt} V(\frac{1}{\sqrt{n}}S_j)$. By Donsker's invariance principle, this converges in distribution to $\int_0^t V(B(s))ds$ where B is a one dimensional Brownian motion. To compute the distribution of $\int_0^t V(B(s))ds$, one tries to compute the Laplace transform $E\left[\exp\left(-\lambda \int_0^t V(B(s))ds\right)\right]$ and then invert to get the desired distribution. Mark Kac [1] showed how to calculate this distribution by solving a differential equation satisfied by the double Laplace transform of $\int_0^t V(B(s))ds$ (the second Laplace transform is taken with respect to t). The answer, called Levy's arcsin law, is stunning for both the fact that the result defies intuition and for the power behind the technique. By the way,

$$P\left(\int_0^1 V(B(s))ds \leq x\right) = \int_0^x \frac{2}{\pi} \arcsin \sqrt{y} dy.$$

This says, more or less, that the prospect of Brownian motion being above 0 half the time is least likely.

The Feynman-Kac formula has a translation into the language of stochastic processes. Let B be a d-dimensional Brownian motion under the measure P_x on the space of continuous paths $C([0.\infty), \mathbb{R}^d)$ so that $P_x(B(0) = x) = 1$. Given suitable functions u, V on $\mathbb{R}^d$, one might express the Feynman-Kac formula by saying

$$\begin{aligned} (1) \qquad & \exp\left(\int_0^t V(B(s))ds\right) u(B(t)) \\ & - \int_0^t \left(\frac{1}{2}\Delta + V\right) u(B(s)) \exp\left(\int_0^s V(B(r))dr\right) ds \end{aligned}$$

is a local martingale (the analog of a fair game in the sense that the P_x-average of this expression is its time 0 value, namely $u(x)$, when evaluated at the right stopping times). Thus if $D \subset \mathbb{R}^d$ and $\tau = \inf\,(t \geq 0 : B(t) \notin D)$ and if u solves the equation $(\frac{1}{2}\Delta + V)u = 0$ in D, with the boundary condition $u = f$, then taking expectations in (1) at time $t = \tau$ we get another formulation of the Feynman-Kac

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formula, assuming the right hand side is finite:

$$u(x) = E_x\left[\exp\left(\int_0^\tau V(B(s))ds\right) f(B(\tau))\right]. \tag{2}$$

A key point of the book under review is to determine when this representation may be valid. This becomes a matter of determining when $E_x\left[\exp\left(\int_0^\tau V(B(s))ds\right)\right]$ is finite. Think of Brownian motion running around the domain D sampling the values of the function (potential) V and keeping a score $\exp\left(\int_0^\tau V(B(s))ds\right)$. This may conceivably average out to some finite quantity under the measure P_x if x is close to the boundary of D and the singularities of V are hidden in some part of the domain far from x and hidden by many parts of ∂D far from the probing eyes of Brownian motion, and it might average out to infinity when x is chosen near a singularity of V. However, a principal theorem in the book (gauge theorem) asserts that if V is taken from a class of functions which are well adapted to the occupation time measure of Brownian motion in the domain D (the Kato class), then the quantity, called the gauge, $g(x) = E_x\left[\exp\left(\int_0^\tau V(B(s))ds\right)\right]$ is either bounded in D or it is identically infinite in D. That is, with some control on the size of the singularities of V, it is impossible to hide them from Brownian motion. When the gauge is finite, the representation of solutions of $(\frac{1}{2}\Delta + V)u = 0$ in D, with the boundary condition $u = f$ given by (2), is valid.

The second development mentioned above appeared in Doob [2]. Conditional Brownian motion was introduced in order to get a probabilistic version of Fatou's boundary limit theorem for harmonic functions. The Brownian motion is conditioned to exit a domain D in a particular way. This is done by taking the transition density for Brownian motion killed on exiting D, call it $p(t,x,y)$, and a positive harmonic function h in D and defining a new transition density $p^h(t,x,y) = p(t,x,y)h(y)/h(x)$. The diffusion process with transition density $p^h(t,x,y)$ is called variously h-Brownian motion or conditional Brownian motion. When h is the Martin kernel with pole a Martin boundary point ξ, then the h-Brownian motion converges to ξ in the Martin topology at the path lifetime. For those unfamiliar with the Martin boundary, consider Lipschitz domains, and then the Martin boundary is the same as the Euclidean boundary and the Martin kernel is the Poisson kernel. The point ξ to which the Brownian motion is conditioned to converge can also be in D, and then h is taken to be the Green function for D with pole at ξ. For the last two choices of h, the process with transition density $p^h(t,x,y)$ is Brownian motion conditioned to exit D at ξ. Sometimes the transition density for the h-Brownian motion with h taken to be the harmonic function with pole at ξ is denoted $p^\xi(t,x,y)$. The corresponding measure on path space is denoted P_x^ξ, and expectation with respect to this measure is denoted by E_x^ξ. We return now to (2) and introduce the notation $\omega_x(A) = P_x(B(\tau)\varepsilon A)$ for A a Borel subset of ∂D. Then ω_x is the harmonic measure of analysis or the exit distribution of Brownian motion from probability. One may write

$$u(x) = \int_{\partial D} E_x^\xi\left[\exp\left(\int_0^\tau V(B(s))ds\right)\right] f(\xi)\omega(d\xi). \tag{3}$$

Thus, it is entirely natural to ask about properties of $E_x^\xi\left[\exp\left(\int_0^\tau V(B(s))ds\right)\right]$. This quantity is called the conditional gauge. A second principal result in this book (the conditional gauge theorem) asserts that under some smoothness assumption on the domain and assuming the potential is in the Kato class, the conditional gauge is either bounded on $D \times \bar{D}$ or it is identically infinite on $D \times \bar{D}$. Again, one might think that the conditional gauge could be finite if the starting point x and exiting point ξ are close together and far from the singularities of V so that the Brownian motion makes the short x to ξ trip without sampling near the singularities, whereas if these points are taken near the singularities, the resulting conditional gauge would be infinite. This cannot happen; even the conditional Brownian motion will have sufficiently many paths moving all around the domain so that such behavior cannot occur. The finiteness of the conditional gauge has many interesting consequences which can be found within the text.

The book of Chung and Zhao contains simplified arguments, improvements and extensions of recent research on the subject described above. It is a very carefully written account. A graduate student or researcher would gain a good perspective of the interplay between analysis and probability theory from a reading. The Feynman-Kac formula is used in so many contexts, it would be difficult to come up with a comprehensive list. This book represents but one interesting aspect of these many applications.

References

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Michael Cranston
University of Rochester
E-mail address: cran@math.rochester.edu