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Theory of Bergman spaces, by Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, Springer-Verlag, New York, 2000, ix+286 pp., $54.95, ISBN 0-387-98791-6

In a first course in complex analysis, students learn a theorem that states that if an analytic function is zero on a non-discrete set inside a region in the complex plane, then the function must be identically zero. In particular, the values that an analytic function takes in the neighborhood of a single point completely determine the function in the whole region. This, of course, is very useful for proving many other theorems about analytic functions. However, it also presents a challenge when one is trying to construct examples with certain required properties. Unlike in a real analysis setting, one cannot just cut the region up into smaller pieces, construct examples locally, and hope to be able to glue everything back together. Over time many ingenious ways have been developed to deal with this problem. It is a large branch of modern complex analysis that tries to devise means to construct certain classes of analytic functions from real variable type parameters.

If the region is the open unit disc $\mathbb{D}$ and one is interested in bounded analytic functions, then a fully developed theory is available. In fact, this theory extends to cover the Hardy spaces H^p for $0 < p < \infty$. On the other hand, for the larger Bergman spaces A^p of the unit disc many new phenomena occur, new theorems and proofs had to be developed, and some basic questions are still not completely settled. Nevertheless, the past ten years have seen a remarkable number of breakthroughs in this area: perhaps most notably the geometric characterization of sequences of interpolation and sampling; a near closing of the gap between necessary and sufficient conditions for zero sequences of A^p-functions; characterizations of bounded Hankel operators, of compact Hankel operators, and of compact Toeplitz operators; the discovery of contractive zero divisors and an A^p-inner-outer factorization; the relationship between Bergman-inner functions and the biharmonic Green function; and other results concerning the invariant subspace structure of A^p.

In the book under review the authors present certain aspects of these new developments. The main focus is on questions concerning the function theory and the invariant subspace structure of the spaces A^p. Hankel and Toeplitz operators are not discussed.

For $0 < p < \infty$ the Bergman space A^p is defined to be the set of all analytic functions f on the open unit disc $\mathbb{D}$ such that $||f||_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi} < \infty$. Here dA has been used to denote two-dimensional Lebesgue measure. For $p \geq 1$ A^p is a Banach space, while for $0 < p < 1$ A^p is a complete space with translation invariant metric given by $d(f,g) = ||f-g||_{A^p}^p$.

The reader should note that in this review we shall discuss only unweighted Bergman spaces. In the book, the authors consider the standard weighted Bergman spaces A^p_α, $\alpha > -1$ and the growth spaces $A^{-\alpha}$, whenever this is feasible. Furthermore, to get maximum benefit out of reading either the book or this review, the reader should have at least a superficial familiarity with the H^p-function theory; see for example [D], [G], [K].

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One easily checks that for each $0 < p < \infty$, $H^p \subsetneq A^p$. But the "gap" between H^p and A^p is significant: Whereas for every $p > 0$ all H^p-functions have nontangential limits a.e. on $\partial\mathbb{D}$, there are nonzero functions f in A^p such that each Stolz angle with vertex in the unit circle contains infinitely many points z such that $f(z) = 0$. It follows then from Plessner's theorem that the image under f of almost every Stolz angle must be dense in the complex plane. This observation makes it apparent that at least part of the subtlety of the A^p-function theory will be in determining how to measure growth and oscillations of functions near $\partial\mathbb{D}$. This problem is magnified once one realizes that for many basic questions about A^p, the answer must depend on the index p.

In the first chapter of the book results about analytic projections and duality are presented. Chapter 2 contains a nice discussion of the Berezin transform, Bergman Carleson measures, and Bergman BMO-VMO.

The cornerstone of the H^p-function theory is the classical inner-outer factorization. It is a parametrization of all H^p-functions in terms of real variable type parameters: Blaschke sequences, nonnegative singular measures, and certain L^1-functions. Furthermore, variations of these parameters lead to fairly predictable changes of the H^p-norms or the values of the function on or near $\partial\mathbb{D}$. One of the goals of a function theory of Bergman spaces would be to obtain a similarly precise and flexible decomposition for Bergman space functions. Thus, in order to start a further investigation of the function theory of Bergman spaces it is natural to consider the invariant subspace structure of A^p. After all, one of the major applications of the classical inner-outer factorization is in Beurling's characterization of the invariant subspaces of H^2.

It came as a real surprise when Apostol, Bercovici, Foias, and Pearcy showed in the mid 1980s that the Bergman shift $M_\zeta : A^2 \to A^2$, $f \mapsto \zeta f$, $\zeta(z) = z$ is in the class $A_{\aleph_0}$ of universal dilations. The consequences of this fact are numerous, and they are all bad news for those who were hoping for a Beurling-type theorem for A^p. In fact, it turns out that the invariant subspace problem for Hilbert spaces is equivalent to the following question: if $\mathcal{M}$ and $\mathcal{N}$ are two invariant subspaces of A^2 with $\mathcal{M} \subseteq \mathcal{N}$ and $\dim \mathcal{N}/\mathcal{M} > 1$, is there a third invariant subspace lying properly between $\mathcal{M}$ and $\mathcal{N}$?

Now, perhaps, it is unlikely that somebody will find a proof for the invariant subspace problem based on results about the Bergman shift, but this result stresses very clearly that there are interesting structural properties of the Bergman shift (and many related operators) that are still awaiting discovery.

As far as analogues of Beurling's theorem go, it turns out that not all invariant subspaces of A^p are singly generated. Nevertheless, one has the following contractive divisor theorem.

Theorem 1. *Let $0 < p < \infty$, let f be a nonzero function in A^p, let n be the smallest integer with $f^{(n)}(0) \neq 0$, and let $[f]$ denote the closure of the polynomial multiples of f in A^p. $[f]$ is called a singly generated invariant subspace.*

Then there is a unique solution φ to the extremal problem

$$\sup\{Re\ g^{(n)}(0) : g \in [f], ||g||^p_{A^p} \leq 1\}.$$

We have $[f] = [\varphi]$ and $H^p \subseteq \frac{[f]}{\varphi} \subseteq A^p$ with contractive inclusions. In particular,

$$(*) \qquad ||\frac{g}{\varphi}||^p_{A^p} \leq ||g||^p_{A^p}$$

for all $g \in [f]$.

As stated, this theorem is from [ARS1]. Special cases were proved in [H1] and [DKSS1], [DKSS2]. In particular, Hedenmalm discovered the contractive divisor property (*) in the context of zero-based invariant subspaces of A^2.

The functions φ occurring in this theorem are the analogues of the classical inner functions. Because of the inequality (*) they also have been called contractive divisors. They are characterized by $||\varphi||^p_{A^p} = 1$ and $\int_{\mathbb{D}} z^n |\varphi(z)|^p \frac{dA(z)}{\pi} = 0$ for $n = 1, 2, \ldots$. As a corollary to this theorem one obtains an A^p-inner-outer factorization.

The proof of the theorem is based on a beautiful connection of the Bergman inner functions and the biharmonic Green function for the unit disc. This and related results are proved in Chapter 3 of the book. For example, there is Shimorin's analogue of Schur's theorem for A^2: every A^2-inner function can be approximated uniformly on compact subsets of $\mathbb{D}$ by zero divisors corresponding to finite zero sets (the analogues of the finite Blaschke products). In fact, the projections onto the corresponding invariant subspaces converge in the strong operator topology.

What is still lacking is a more transparent description of the A^p-inner and outer functions. What exactly are the analogues of Blaschke products? One says that a sequence $A = \{a_n\}_{n \geq 0} \subseteq \mathbb{D}$ is a zero sequence for A^p if there is a function $f \in A^p$ such that $f(z) = 0$ if and only if z is one of the points in A. Here and in the following we shall always mean that f has a zero of multiplicity m at z if z is repeated exactly m times in the sequence A. It is known that Theorem 1 applies to all zero-based invariant subspaces, i.e. subspaces of the form $I(A) = \{f \in A^p : f(a) = 0 \text{ for all } a \in A\}$ for some zero sequence A of A^p. However, it is still an open question to give computable necessary and sufficient conditions for a sequence A to be an A^p-zero sequence.

The first substantial results about Bergman zero sequences are from the 1970s. Horowitz showed that unlike the H^p situation the conditions for A^p zero sequences depend on p, and for each p there are two zero sequences A and B such that the union $A \cup B$ is not an A^p zero sequence. Furthermore, for every $p \geq 0$, any subsequence of a zero sequence is a zero sequence for A^p.

Of course, every Blaschke sequence is a zero sequence for each A^p. In a certain sense not too many more sequences are allowed.

Theorem 2. *Let* $0 < p < \infty$*, and let* $\{a_n\}_{n \geq 0}$ *be a zero sequence for* A^p*.*

(a) If $\varepsilon > 0$*, then*

$$\sum_{n \geq 0} \frac{1 - |a_n|}{\left(\log \frac{1}{1-|a_n|}\right)^{1+\varepsilon}} < \infty.$$

(b) Any subset of all points of $\{a_n\}_{n \geq 0}$ *that lies on a single radius or in a single nontangential approach region (a Stolz angle) must satisfy the Blaschke condition.*

Like the Blaschke condition, this theorem follows from Jensen's formula. Part (b) together with the fact that Blaschke sequences are not the only zero sequences shows that any necessary and sufficient condition for A^p zero sequences must take the angular distribution of the zeros into account.

(b) is based on the following observation: if $f \in A^p$, then the function h, $h(z) = (1-z)^{4/p} f(z)$ has the same zeros as f and is bounded in a disc centered at $1/2$ and tangent to the unit circle at 1. Thus, the zeros of h will satisfy a Blaschke condition with respect to this disc, and this easily leads to the statement about the zeros of f.

Of course, the conclusion of this last observation holds for unions of finitely many Stolz regions, and a very careful quantitative analysis of the previous argument leads to a necessary condition for a sequence $\{a_n\}_{n\geq 0}$ to be an A^p-zero set that is almost sufficient.

Let F be a finite subset of the unit circle $\partial\mathbb{D}$ with complementary arcs $\{I_n\}_n$; then the Beurling-Carleson characteristic of F is defined to be $\hat{\kappa}(F) = \sum_n |I_n|_s \log \frac{e}{|I_n|_s}$. Here $|I|_s$ was used to denote the normalized arclength of the arc I. The *upper asymptotic κ-density* of a sequence A is defined by

$$D^+(A) = \limsup_{\hat{\kappa}(F)\to\infty} \frac{\sum(A, r_F)}{\hat{\kappa}(F)}.$$

Here r_F denotes the union of radii with endpoints in F, and $\sum(A, r_F)$ is a Blaschke-type counting function defined by $\sum(A, r_F) = \frac{1}{2}\sum_{a\in A\cap r_F}(1-|a|^2)$. It is notable that the quantity $D^+(A)$ remains unchanged if instead of using r_F, one counted all zeros in a "Stolz-star" s_F with vertices in F.

Theorem 3. *Let $0 < p < \infty$ and let A be a sequence in $\mathbb{D}$.*

If $D^+(A) < \frac{1}{p}$, then A is an A^p-zero sequence. If $D^+(A) > \frac{1}{p}$, then A is not a A^p-zero sequence.

Zero sets are discussed in Chapter 4 of the book. The original proofs of the necessary conditions in Theorem 3 were based on the idea as in the proof of Theorem 2(b) and some precise knowledge about conformal mappings on almost circular domains [K1], [S]. For the current book, this part of the proof has been modified and involves elegant estimates for the growth of harmonic functions and a very general maximum principle which applies to a situation where the functions have a finite number of singularities on the boundary of the region in question. This is a subtle point of which the authors were undoubtedly aware. However, in the exposition it is glossed over, and some care needs to be taken when reading the proof of Theorem 4.23.

The sufficiency of the conditions of Theorem 3 is established with a normal families argument, and the construction does not provide an expression for the limit. Each element of the approximating sequence depends on a set of parameters. These parameters are the solutions to certain linear programming-type extremal problems.

Although Theorem 3 does not succeed in completely describing the A^p-zero sets, the estimates one obtains are good enough to lead to a description of the A^p-interpolation and sampling sequences. These concepts are useful for a discretization of A^p. Their geometric characterizations were found by Seip.

Every function $f \in A^p$ satisfies the growth estimate $(1-|z|^2)^{2/p}|f(z)| = o(1)$ as $|z| \to 1$. Thus, if $A = \{a_n\}_{n\geq 0}$ is a sequence of points in $\mathbb{D}$, then for each $0 < p < \infty$ the linear map $T_p : f \mapsto \{(1-|a_n|^2)^{2/p} f(a_n)\}_{n\geq 0}$ maps A^p into c_0. The sequence A is called *interpolating* for A^p if T_p takes A^p into and onto l^p. Actually, if one just requires that T_p is onto, then the boundedness follows automatically.

One quickly checks that the class of A^p-interpolating sequences is invariant under Moebius transformations of the unit disc and that interpolating sequences must be zero sequences, although the converse is not true.

One is thus led to define the Moebius invariant *uniform upper asymptotic κ-density* of a sequence $A = \{a_n\}_{n\geq 0}$ by

$$D_u^+(A) = \limsup_{\hat{\kappa}(F)\to\infty} \sup_n \frac{\Lambda(A_n, r_F)}{\hat{\kappa}(F)}.$$

Here A_n denotes the image of $A\backslash\{a_n\}$ under the Moebius transformation that sends a_n to 0, and $\Lambda(A, r_F)$ is a counting function defined by $\Lambda(A, r_F) = \sum_{a\in A\cap r_F} \log \frac{1}{|a|}$. Note the close relationship between the counting functions $\sum(A, r_F)$ and $\Lambda(A, r_F)$. In fact, if $0 \notin A$, then one could have used $\Lambda(A, r_F)$ in the definition of $D^+(A)$ without changing its value.

Theorem 4. *Let $0 < p < \infty$. A sequence A in $\mathbb{D}$ is interpolating for A^p if and only if $D_u^+(A) < \frac{1}{p}$.*

The Moebius-invariance and the zero set results quickly imply that any A^p-interpolating sequence A must satisfy $D_u^+(A) \leq \frac{1}{p}$. To get strict inequality, one must also observe that sufficiently small perturbations (in the pseudohyperbolic Hausdorff metric) of interpolating sequences are interpolating and that one can change the quantity D_u^+ with such perturbations.

Theorem 4 and the characterization of sampling sequences are done in Chapter 5 of the book. Sampling is a concept that has no analogue in the H^p-theory, except for the dominating sequences in the H^∞-context. A sequence A is a *sampling sequence* for A^p if the operator T_p maps A^p into l^p and is bounded below. Thus, with a sampling sequence, one obtains a discretely supported norm that is equivalent to the A^p-norm. It is notable that apparently current knowledge permits only the characterization of sampling sequences for $p \geq 1$.

One might complain that for given sequences A the density $D_u^+(A)$ is not very easy to compute. Thus, considerable effort goes into showing that for separated sequences (in the pseudohyperbolic metric), the density $D_u^+(A)$ equals yet another density, which in the book is called the upper Seip density $D_s^+(A)$. Since interpolating sequences must be separated, this gives another characterization of interpolating sequences. The lower Seip density $D_s^-(A)$ is used in the description of sampling sequences. In the final section of Chapter 5, these densities are computed for certain regularly distributed sequences.

Chapter 6 is a short discussion of some further topics about invariant subspaces in A^p. By use of sampling and interpolating sequences, examples of nonsingly generated invariant subspaces are given; see [H2]. Such subspaces are difficult to exhibit, because all nonzero functions in such a subspace must be irregular near almost every point of $\partial\mathbb{D}$; see [ARS2]. Furthermore, Shimorin's recent new proof of the wandering subspace theorem for A^2 is presented. This theorem is an analogue of Theorem 1, which is valid for arbitrary invariant subspaces of A^2; see [ARS1] or [MR] for yet another new proof.

A function $f \in A^p$ is called cyclic if the polynomial multiples of f are dense in A^p. It follows from Theorem 1 that the cyclic functions are the A^p-outer functions, but it still is an open problem to describe cyclic functions in terms of their size near $\partial\mathbb{D}$. The gap in the current knowledge here is comparable to the gap between the conditions in Theorem 3. The following theorem is proved in Chapter 7, and it has been known since the 1970s [K2], [BK]. By use of the contractive divisor property of Theorem 1, one can now give a shorter proof of one part of it.

Theorem 5. *Let $0 < p < \infty$, and let $f \in A^q$, $q > p$. Then f is cyclic in A^p if and only if f has no zeros in $\mathbb{D}$ and if the "premeasure" associated with f is "κ-absolutely continuous".*

The concepts "premeasure" and "κ-absolutely continuous" have their origins in Korenblum's "Bergman-appropriate" generalization of the classical Herglotz theorem. For example, absolutely continuous (w.r.t. Lebesgue measure) measures are κ-absolutely continuous. Singular measures are κ-absolutely continuous if and only if they place no mass on closed sets $F \subseteq \partial\mathbb{D}$ of Lebesgue measure zero and with finite Beurling-Carleson characteristic $\hat{\kappa}(F)$. This leads to a characterization of which classical singular inner functions are cyclic in A^p.

Chapter 8 is based entirely on a paper by Borichev and Hedenmalm in which it is shown that Theorem 5 becomes false if the hypothesis $q > p$ is replaced with $q = p$.

Chapter 9 on logarithmically subharmonic weighted Bergman spaces contains results that were announced in [HJS]. The authors obtain a strengthening of the contractive divisor property of Theorem 1. In particular, if $A \subseteq B$ are two finite sequences in $\mathbb{D}$, and if φ_A and φ_B are the contractive divisors associated with $I(A)$ and $I(B)$, then $||\varphi_A f||^p_{A^p} \leq ||\varphi_B f||^p_{A^p}$ for all $f \in A^p$. It follows that

$$H^p \subseteq \frac{I(B)}{\varphi_B} \subseteq \frac{I(A)}{\varphi_A} \subseteq A^p.$$

Also, a special case of the results here implies that if φ is any A^2-inner function, then the Hilbert space of analytic functions $\mathcal{H} = \frac{[\varphi]}{\varphi}$ has a reproducing kernel of the form $k_w(z) = \frac{1-\overline{w}zl_w(z)}{(1-\overline{w}z)^2}$ where $l_w(z)$ is a positive definite kernel in $\mathbb{D}$. These are beautiful results and they have their place in this book. However, for the first of these, the authors had to break the "self-containment" promise which is made on the backcover of the book. Without proof, they quoted and used a theorem from [HS].

At the end of each chapter the book has a short section with notes and then some exercises. Typically, the notes contain a brief listing of the sources for theorems and proofs, and occasionally there is a mention of related results not covered in the book. Also, many of the exercises are really theorems whose proofs did not make it into the book and whose references are provided.

This is a rough overview of the topics covered in the book. After Chapter 2 it would have been natural to include some material on Hankel and Toeplitz operators, but one can understand the authors' choice of topics. In the first two chapters there is some overlap with Zhu's book [Z], and some results about Bergman zero sets also appear in Djrbashian and Shamoian's book [DS], but the current treatment is either more general or more complete. In fact, most of the material covered has never appeared in book form before. Given the choice of topics, the authors have made a considerable effort to include most state-of-the-art theorems with complete proofs. The material is extremely well-organized, and although the explanations for particular details are sometimes brief, the flow of ideas becomes very clear in the presentation. It was pointed out by Rachel Weir that there is a problem with the case $p = 1$ in Proposition 3.5 (existence), but this reviewer is not aware of any other mistakes. Researchers in the field will be glad to have a reliable reference book available.

The book should be accessible to mid- to upper-level graduate students, and it would be suitable as a resource for a topics course.

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