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Dynamical systems and semisimple groups, an introduction, by Renato Feres, Cambridge Univ. Press, Cambridge, 1998, xvi+245 pp., $54.95, ISBN 0-521-59162-7

To many mathematicians, the phrase "dynamical systems" refers mainly to transformations and flows on smooth manifolds (read "smooth dynamics"). However, over time, the meaning has grown to include transformations and flows on topological spaces (read "topological dynamics") and on measure spaces (read "ergodic theory"). Of course, in many situations, one might have more than one transformation or more than one flow. If n invertible transformations on a space commute, then one obtains an action of $\mathbb{Z}^n$ on the space. Similarly, if n invertible complete flows commute, then one has an action of $\mathbb{R}^n$.

In some situations, however, the transformations and flows may *not* commute, in which case one obtains an action of a more complicated group. Generally speaking, as the group under consideration attains higher levels of complexity, the possibilities for actions become fewer, since one must find transformations and flows that satisfy more and more relations. One finds that one does not always have great flexibility in choosing actions; this lack of flexibility is often called "rigidity".

From a certain perspective, terminology notwithstanding, the most complicated Lie groups are simple Lie groups. One may hope to classify how these groups may act on a variety of geometric spaces. This is R. J. Zimmer's program. The end goal of the book under review is to give

- an accessible exposition of Zimmer's cocycle superrigidity theorem, which is an ergodic theoretic (*i.e.*, measure theoretic) result; and
- to give a new proof that works, not just in the measure theoretic category, but in the smooth category.

The main result, Theorem 10.4.1, p. 212, appeared previously as joint work of Feres and F. Labourie. (See [FL97].)

Below I wish to give some motivation for superrigidity in general. In the second section, I will comment on the book at hand.

1. Superrigidity for lattices and cocycles

If G is a locally compact topological group and H is a closed subgroup of G, then we say that H is **cocompact** in G if the quotient topology on G/H is compact. We say that H is of **cofinite volume** in G if there is a finite G-invariant Borel measure on G/H. We say that H is a **lattice** in G if H is a discrete, cofinite volume subgroup of G. A discrete, cocompact subgroup is of cofinite volume, and is therefore a (cocompact) lattice.

1a. Superrigidity and arithmeticity of lattices.

Closed Riemann surfaces are classified: The universal cover of a Riemann surface is the plane, the sphere or the disk, and, in each of these three cases, we know the possible discrete cocompact groups of conformal automorphisms, up to conjugation.

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There are a variety of natural ways to generalize the term "Riemann surface". One way is to view a (uniformized, higher genus) Riemann surface as a locally symmetric space modeled on the group $\mathrm{SL}_2(\mathbb{R})$. That is, for each discrete, cocompact, torsion-free subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, one obtains the uniformized Riemann surface $\Gamma\backslash(\mathrm{SL}_2(\mathbb{R}))/(\mathrm{SO}(2))$. All higher genus closed Riemann surfaces can be constructed in this way. For other simple Lie groups, like $\mathrm{SL}_3(\mathbb{R})$, one may ask a similar question: What are the closed, locally symmetric spaces modeled on $\mathrm{SL}_3(\mathbb{R})$? That is, what are the discrete, cocompact, torsion-free subgroups of $\mathrm{SL}_3(\mathbb{R})$? For each such group Γ, one obtains a locally symmetric space $\Gamma\backslash(\mathrm{SL}_3(\mathbb{R}))/(\mathrm{SO}(3))$.

Riemann surfaces with finite topology and no big ends are classified: They are in one-to-one correspondence with discrete, cofinite volume, torsion-free subgroups of $\mathrm{SL}_2(\mathbb{R})$. By analogy, one may ask to find all discrete, cofinite volume, torsion-free subgroups of $\mathrm{SL}_3(\mathbb{R})$, thereby classifying all finite volume locally symmetric spaces modeled on $\mathrm{SL}_3(\mathbb{R})$. One may also ask these questions for other simple and semisimple Lie groups besides $\mathrm{SL}_3(\mathbb{R})$, but we will, in this review, mostly hold to $\mathrm{SL}_3(\mathbb{R})$, for the sake of concreteness.

With the success of Teichmüller Theory in studying $\mathrm{SL}_2(\mathbb{R})$, one might expect that similar methods could be made to work to analyze other groups like $\mathrm{SL}_3(\mathbb{R})$. However, we now know that $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{R})$ are so different that this cannot work. For example, there are uncountably many closed Riemann surfaces, but it turns out that there are, up to isomorphism, only countably many compact locally symmetric spaces modeled on $\mathrm{SL}_3(\mathbb{R})$. One does not have as much flexibility in forming spaces modeled on $\mathrm{SL}_3(\mathbb{R})$ as on $\mathrm{SL}_2(\mathbb{R})$. This lack of flexibility is, again, called "rigidity". Over time, entirely new techniques were found to handle groups like $\mathrm{SL}_3(\mathbb{R})$, and the most powerful of these techniques is "superrigidity/arithmeticity", due to G. A. Margulis.

For both $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{R})$, there is a collection of lattices that are constructed and studied through number theory, and these lattices are called "arithmetic lattices", which we now define. (*Note:* The noun arithmetic is pronounced "aRITHmetic", but the adjective arithemetic is pronounced "arithMEtic". The noun arithmeticity is pronounced "arithmeTIcity".)

For any integer $N \geq 1$, a subgroup H of $\mathrm{GL}_N(\mathbb{R})$ is **$\mathbb{Z}$-algebraic** if it is the zero locus of a collection of polynomials in the N^2 matrix entries, with each of these polynomials having coefficients in $\mathbb{Z}$. For example, $\mathrm{SL}_N(\mathbb{R})$ is $\mathbb{Z}$-algebraic, being the zero locus of the polynomial $1 - \det$. A $\mathbb{Z}$-algebraic subgroup H of $\mathrm{GL}_N(\mathbb{R})$ is closed and is therefore a Lie subgroup of $\mathrm{GL}_N(\mathbb{R})$; its identity component will be denoted by H°. It can happen that H° is not $\mathbb{Z}$-algebraic, even if H is; however, H° is always a connected Lie group.

A lattice Γ in a Lie group G is **arithmetic** if, for some integer $N \geq 1$, there is a $\mathbb{Z}$-algebraic $H \subseteq \mathrm{GL}_N(\mathbb{R})$ and a Lie group isomorphism $\iota : G \to H^\circ$ such that $\Gamma = \iota^{-1}(\mathrm{GL}_N(\mathbb{Z}))$. The isomorphism ι is sometimes called a **$\mathbb{Z}$-structure** on G. Given a $\mathbb{Z}$-structure $\iota : G \to \mathrm{GL}_N(\mathbb{R})$, let us agree to say that its **$\mathbb{Z}$-points** are $\iota^{-1}(\mathrm{GL}_N(\mathbb{Z}))$. With this terminology, a lattice is arithmetic iff it is the $\mathbb{Z}$-points of some $\mathbb{Z}$-structure.

Given two discrete subgroups of a Lie group, we say that they are **commensurable** if their intersection has finite index in each of them. If so, and if one of them is a cocompact (resp. non-cocompact) lattice, then the other must be as well. On the geometric side, one might say that two locally symmetric spaces are commensurable if they have a common finite cover. In some sense, commensurable lattices are 'almost' the same, as are commensurable locally symmetric spaces.

Taking this into account, we modify our goal: we now aim to find all commensurability classes of torsion-free lattices (cocompact and non-cocompact) in $\mathrm{SL}_3(\mathbb{R})$. By a well-known result (often called "Selberg's Lemma"), any lattice in $\mathrm{SL}_3(\mathbb{R})$ has a subgroup of finite index that is torsion-free, so every commensurability class of lattices in $\mathrm{SL}_3(\mathbb{R})$ contains a torsion-free lattice. Thus our goal is simply to find all commensurability classes of lattices in $\mathrm{SL}_3(\mathbb{R})$.

The phenomenal achievement of superrigidity/arithmeticity is to show that, for many simple Lie groups (including $\mathrm{SL}_3(\mathbb{R})$), every non-cocompact lattice is commensurable with an arithmetic lattice. We remark that a theorem of Borel and Harish-Chandra [BHC62] asserts that, if Γ is the $\mathbb{Z}$-points of a $\mathbb{Z}$-structure on a connected semisimple Lie group G, then Γ is a lattice in G. Moreover, if G is a noncompact connected simple Lie group (like $\mathrm{SL}_3(\mathbb{R})$) and if Γ is the $\mathbb{Z}$-points of a $\mathbb{Z}$-structure on G, then Γ cannot be cocompact. Thus the problem of finding non-cocompact lattices in $\mathrm{SL}_3(\mathbb{R})$ is reduced to the algebraic problem of finding all $\mathbb{Z}$-structures on $\mathrm{SL}_3(\mathbb{R})$.

There are many methods for finding $\mathbb{Z}$-structures in semisimple Lie groups, and there is even a cohomology theory (Galois cohomology), one of whose applications is classification of $\mathbb{Z}$-structures. I know very little of this, but I will take the point of view that, for any connected semisimple Lie group, we "know" all of its $\mathbb{Z}$-structures. Assuming this, the non-cocompact lattices in $\mathrm{SL}_3(\mathbb{R})$ are classified, up to commensurability.

For cocompact lattices, the situation is only slightly more difficult: Let G be a connected semisimple Lie group and let Γ be a lattice in G, possibly cocompact. Let us agree that Γ is **projected arithmetic** if there is a connected semisimple Lie group $\widehat{G}$, a surjective Lie group homomorphism $\pi : \widehat{G} \to G$ and an arithmetic lattice $\widehat{\Gamma}$ in $\widehat{G}$ such that $\ker(\pi)$ is compact and such that $\Gamma = \pi(\widehat{\Gamma})$. We remark that, if $\pi : \widehat{G} \to G$ is a surjective homomorphism of Lie groups, if $\ker(\pi)$ is compact and if $\widehat{\Gamma}$ is a cocompact (resp. non-cocompact) lattice in $\widehat{G}$, then $\pi(\widehat{\Gamma})$ is a cocompact (resp. non-cocompact) lattice in G.

From work of Margulis, every lattice in $\mathrm{SL}_3(\mathbb{R})$ is commensurable with one that is projected arithmetic. This arithmeticity result extends to all semisimple Lie groups of real rank ≥ 2, at which level of generality we will call it Margulis' **Arithmeticity Theorem for Lattices**. For an exposition, one may consult [Ma91] or [Z84, Theorem 6.1.2, p. 114, and Corollary 6.1.10, p. 121]. In any case, from these results, we may proceed from a classification of $\mathbb{Z}$-structures on connected semisimple Lie groups to a classification of lattices in $\mathrm{SL}_3(\mathbb{R})$, and, in fact, on any semisimple Lie group of real rank ≥ 2. So, assuming such $\mathbb{Z}$-structures are classified, we then know all lattices in, say, $\mathrm{SL}_3(\mathbb{R})$. This is, then, the $\mathrm{SL}_3(\mathbb{R})$ analogue to Teichmüller theory.

Some authors wish to replace our cumbersome "any lattice in $\mathrm{SL}_3(\mathbb{R})$ is commensurable with one which is projected arithmetic" with the simpler phrase "all lattices in $\mathrm{SL}_3(\mathbb{R})$ are arithmetic", but this requires a different definition of the word "arithmetic" than ours. In fact, one will find, in the literature, a number of different definitions of the phrase "arithmetic lattice". For this review, let us agree that a lattice is **nearly arithmetic** if it is commensurable with a projected arithmetic lattice. Then we may say that every lattice in $\mathrm{SL}_3(\mathbb{R})$ is nearly arithmetic. (The definition of "arithmetic" in [Z84] is exactly what we here call "almost arithmetic".)

If $f : A \to B$ is a group homomorphism, then, by a **finite index restriction** of f, we simply mean the restriction of f to some finite index subgroup of A. For any rational prime p, let $\mathbb{Q}_p$ denote the topological field of p-adic numbers.

A topological field k is **local** if it is locally compact and not discrete. Such fields are essentially classified. (See [W74, Chapter I].) In characteristic zero, the only examples are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$ and finite extensions of $\mathbb{Q}_p$. The fields $\mathbb{R}$ and $\mathbb{C}$ are **archimedean**, while $\mathbb{Q}_p$ and its finite extensions are **non-archimedean**. By **local representation** we will mean a finite-dimensional representation of a group over a local field of characteristic zero. By an **archimedean local representation** (resp. **non-archimedean local representation**), we will mean a finite-dimensional representation of a group over an archimedean (resp. non-archimedean) local field of characteristic zero. Let Γ be a lattice in a connected Lie group G, and let $\rho : \Gamma \to \mathrm{GL}_d(k)$ be a local representation of Γ. We will say that ρ **extends** (or **is extendable**) if there is a representation $\mathrm{R} : G \to \mathrm{GL}_d(k)$ such that $\rho = \mathrm{R}\,|\Gamma$.

Let Γ be a nearly arithmetic lattice in a connected semisimple Lie group G. We will say that an archimedean local representation $\rho : \Gamma \to \mathrm{GL}_d(k)$ is **nearly arithmetic** if there are

- (•) a connected semisimple Lie group $\widehat{G}$, and
- (•) a surjective Lie group homomorphism $\pi : \widehat{G} \to G$

such that

- (•) $\ker(\pi)$ is compact, and
- (•) some finite index restriction of $\rho \circ \pi : \pi^{-1}(\Gamma) \to \mathrm{GL}_d(k)$ extends.

Note that $\pi^{-1}(\Gamma) \subseteq \widehat{G}$, so $\rho \circ \pi : \pi^{-1}(\Gamma) \to \mathrm{GL}_d(k)$ will extend not to G, but, rather, to $\widehat{G}$. As $\widehat{G}$ is semisimple, the local representations of $\widehat{G}$ are classified, so we therefore nearly obtain a classification of nearly arithmetic representations of nearly arithmetic lattices; they are the ones that, up to finite index, are obtained via arithmetic constructions.

To understand why, in the definition above, it is important to allow for a compact extension $\widehat{G}$ of G, consider the following example: Fix an integer $d \geq 3$, let G be the identity component of $\mathrm{SO}(d-1,1)$ and let $K := \mathrm{SO}(d)$. Let $\widehat{G} := G \times K$. Let $p : \widehat{G} \to G$ and $q : \widehat{G} \to K$ be the first and second coordinate projections. Following [Z84, Example 6.1.5, p. 117], it is possible (using restriction of scalars) to construct a $\mathbb{Z}$-structure on $\widehat{G}$ whose $\mathbb{Z}$-points $\widehat{\Gamma}$ satisfy: $\widehat{\Gamma} \cap (\ker(p)) = \{1\}$. Let $\Gamma := p(\widehat{\Gamma})$. Then $p|\widehat{\Gamma} : \widehat{\Gamma} \to \Gamma$ is an isomorphism.

The representation $\rho := q \circ ((p|\widehat{\Gamma})^{-1}) : \Gamma \to K \subseteq \mathrm{GL}_d(\mathbb{R})$ is nearly extendable and its image is contained in the compact group K. It is well-known that G has no nontrivial local representations whose image is contained in a compact matrix group, so ρ does not extend. My point of view on this example is that even though, technically speaking, Γ lives in G, it has an avatar that lives in $\widehat{G}$ and this avatar comes with a projection map to K which is set up to extend to $\widehat{G}$, but not to G.

Margulis proved not only that any lattice Γ in $\mathrm{SL}_3(\mathbb{R})$ is nearly arithmetic, he also proved near arithmeticity of any archimedean local representation of Γ. He also proved a similar result for non-archimedean representations, in which the group $\widehat{G}$ might be a product of (real) semisimple Lie groups and p-adic semisimple Lie groups.

In any case, the sense of these results is that we have a good understanding, up to passing to subgroups of finite index, of all the local representations of any lattice in a wide variety of simple and semisimple Lie groups, including $\mathrm{SL}_3(\mathbb{R})$. We will call this result Margulis' **Arithmeticity Theorem for Representations**.

If one assumes the Arithmeticity Theorems for Lattices and for Representations, then one has such a good picture of the representation theory of lattices that it becomes relatively easy to prove "superrigidity", whose philosophy is that, if a local representation of a lattice in, say, $\mathrm{SL}_3(\mathbb{R})$ does not extend, then it must be because there is some compact group that is "getting in the way". This can be made precise in a number of ways. My favorite is: Let Γ be a lattice in $\mathrm{SL}_3(\mathbb{R})$ and let $\rho : \Gamma \to \mathrm{GL}_d(k)$ be a local representation of Γ. Then both of the following are true:

(1) ρ is completely reducible; and
(2) if ρ is irreducible, then either $\rho(\Gamma)$ has compact closure in $\mathrm{GL}_d(k)$, or some finite index restriction of ρ extends.

Both (1) and (2) are important results and both are due to Margulis. As usual, (2) is not special to $\mathrm{SL}_3(\mathbb{R})$; it generalizes to semisimple Lie groups of real rank ≥ 2. We will refer to this generalization as Margulis' **Superrigidity Theorem**. Again, depending on whether k is archimedean or not, we have either archimedean or non-archimedean superrigidity. In (2), one might hope to replace "some finite index restriction of ρ extends" by "ρ extends", but, unless one assumes that the Zariski closure of $\rho(\Gamma)$ is center-free, the resulting statement would not be true; see the paragraph following Corollary 24.2 on pp. 189–190 of [Mo73].

As I mentioned, superrigidity follows from arithmeticity. It was Margulis' revolutionary insight to realize that one could turn the logic around and prove arithmeticity from superrigidity. Moreover, he also discovered that one could prove the Superrigidity Theorem through an exceedingly clever collection of dynamical tricks. (See [Ma91] and [Z84].)

Here is a very broad outline of the proof that superrigidity implies arithmeticity: One feels that a lattice is arithmetic or nearly arithmetic if it (or a finite index subgroup of it) has some incarnation as matrices with integer entries. Using *archimedean* superrigidity, Margulis showed that any lattice in $\mathrm{SL}_3(\mathbb{R})$ has the property that the traces of its elements are in some **number field**, *i.e.*, some finite-dimensional extension of $\mathbb{Q}$. A standard representation theoretic trick then provides an incarnation of the lattice as matrices with entries in that number field. Via a process called restriction of scalars, this number field can be assumed to be $\mathbb{Q}$. That is, the matrices, in some incarnation, have rational entries.

For any rational prime p, let $\mathbb{Z}_p$ denote the commutative ring of p-adic integers. An elementary "local-global" result states that, for all $x \in \mathbb{Q}$, we have:

$$x \in \mathbb{Z} \qquad \text{iff} \quad \forall \text{ rational primes } p,\ x \in \mathbb{Z}_p.$$

That is, a rational number is an integer iff it is "locally an integer". Using *non-archimedean* superrigidity, Margulis showed that, after passing to a subgroup of finite index, the entries of the various matrices were all locally integral, and therefore integral.

This sketch omits a great deal, but the point I wish to stress here is that both the archimedean and non-archimedean forms of superrigidity are needed to prove arithmeticity. For more details, see §6.1, pp. 114–122 of [Z84].

1b. **The Mackey-Zimmer Program.** Let a group G act on a set S and let H be a group. A map $\alpha : G \times S \to H$ is a

cocycle from the G-action on S, with values in H

if, for all $g, g' \in G$, for all $s \in S$, we have $[\alpha(g', gs)][\alpha(g, s)] = \alpha(g'g, s)$. Two cocycles $\alpha, \beta : G \times S \to H$ are **equivalent** if there is a map $\eta : S \to H$ such that, for all $g \in G$, for all $s \in S$, we have $[\eta(gs)][\alpha(g, s)] = [\beta(g, s)][\eta(s)]$.

Let Γ be a subgroup of G. Then G acts transitively on G/Γ in the usual way. Say that two homomorphisms $a, b : \Gamma \to H$ are **equivalent** or **conjugate** if, for some $h \in H$, for all $g \in \Gamma$, we have $h[a(g)] = [b(g)]h$. It is then an interesting exercise to establish that there is a one-to-one correspondence between equivalence classes of homomorphisms $\Gamma \to H$ and equivalence classes of cocycles $G \times (G/\Gamma) \to H$.

Thus one may think of matrix representations of Γ as cocycles on a transitive group action with values in a matrix group. Part of G. W. Mackey's program, as interpreted by Zimmer, is to use this elementary fact to take representation theoretic statements and find dynamical analogues for them. If one takes the resulting dynamical statement and restricts to transitive actions, the result is equivalent to the original representation theoretic statement. If the representation theoretic statement is known to be true, one hopes that the dynamical analogue will also be true, and that a similar proof will work.

Since Margulis' proof of the Superrigidity Theorem is dynamical in nature, it was Zimmer's hope to adapt it to prove a corresponding result about cocycles. This he did, producing his **Cocycle Superrigidity Theorem** [Z84, Theorem 5.2.5, p. 98], a theorem that contains Margulis' Superrigidity Theorem as its transitive special case. Cocycle superrigidity has had a great many diverse and sometimes surprising applications. For example, its S-arithmetic form [Z84, Theorem 10.1.6, p. 189] has recently settled a longstanding problem in logic [AK00].

2. Review

This is an excellent book, and I recommend it highly.

The first nine chapters skillfully develop necessary background material needed in the proof of cocycle superrigidity. In the tenth appears its statement and proof.

2a. **Review of background material, Chapters 1–9.** In Chapters 1–9, the proofs of many standard results are done with great skill and efficiency. These include

- the Frobenius Theorem (Theorem 3.2.1, p. 39);
- Whitney's Theorem on the almost connectedness of real algebraic subsets of Euclidean space (Proposition 4.7.1, p. 68);
- Oseledec' Theorem (Chapter 9, pp. 174–195);
- the Birkhoff Ergodic Theorem (Theorem 8.4.1(2), p. 162);
- the Moore Ergodicity Theorem (Theorem 8.3.1, p. 156);
- basic results on Anosov diffeomorphisms (Section 1.5, pp. 12–16, and Section 8.5, pp. 165–166);

- basics of real algebraic geometry vs. complex algebraic geometry (see, for example, Proposition 4.6.2, p. 67);
- basics of semisimple Lie groups, both real and complex, with a description of the classical groups (Chapter 5, pp. 77–87);
- the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ in §7.7, pp. 139–143;
- the Borel Density Theorem (Theorem 4.9.7, p. 74); and
- Weyl's Unitary Trick (p. 139).

In addition, I would also highlight the section on integrating infinitesimal actions (Section 3.9, pp. 52–58). The results in that section may be described as folklore, and have been begging to appear, with careful proof, in a standard modern reference book.

To all who work in this area, the lack of a good basic reference book on real semisimple groups has been a problem. The book under review should partially alleviate this situation.

2b. **Review of the chapter on Cocycle Superrigidity, Chapter 10.** To me, this is the gem of the book.

Zimmer's original Cocycle Superrigidity Theorem was essentially a theorem in ergodic theory, part of which classified, up to measure theory, Zariski dense cocycles from

> ergodic, finite measure preserving actions of higher rank simple algebraic groups on measure spaces,

with values in noncompact simple real algebraic groups. Here, "Zariski dense" means that the cocycle is not equivalent to a cocycle with values in a proper real algebraic subgroup.

To understand Feres' point of view *vis-a-vis* cocycle superrigidity, it helps to know that one way in which cocycles arise is in studying automorphisms of principal bundles: Let H be a Lie group, let M be a manifold and let $\xi : P \to M$ be a principal H-bundle over M. Let a group G act by principal bundle automorphisms on P. (The G-action on P covers a possibly nontrivial G-action on M.) Let $\Phi : P \to M \times H$ be a measurable trivialization of P over M. Then, under Φ, the action of G on P corresponds to an action of G on $M \times H$, and there exists a unique cocycle $\alpha : G \times M \to H$ such that, for all $q = (m, h) \in M \times H$, we have $gq = (gm, (\alpha(g, m))h)$.

A different measurable trivialization gives rise to a different cocycle; however, the new cocycle will be equivalent to α. Thus an action on a principal bundle gives rise to an equivalence class of cocycles. Now assume that G is a higher rank, simple, real algebraic group. Assume that the action of G on M preserves a Borel measure on M and that the G-action is ergodic with respect to this measure. If the group H under consideration is also a simple real algebraic group, then one can hope to prove that the cocycles obtained from measurable trivializations are Zariski dense, in which case the bundle action is, from a measure-theoretic point of view, completely understood via cocycle superrigidity.

Since actions of groups on geometric spaces give rise to many kinds of principal bundle actions, it is not surprising that one of the main uses of cocycle superrigidity is found in differential geometric dynamics. The bundles obtained in geometry are always trivial from a measure theoretic point of view. That is, one can always find a measurable trivialization of any principal bundle. However, geometers are usually interested in viewing principal bundles as smooth bundles, and Zimmer's original cocycle superrigidity theorem does not take the smooth structure into account.

Theorem 10.4.1, p. 212, is the climax of the book, and I will call it here the **Principal Bundle Superrigidity Theorem**. Restricting this result to measurable principal bundles, one obtains the archimedean part of Zimmer's cocycle superrigidity theorem [Z84, Theorem 5.2.5, p. 98]. However, in the Principal Bundle Superrigidity Theorem, everything is phrased entirely in terms of principal bundles; the word "cocycle" does not appear. Moreover, as one might hope, this theorem applies also to C^r bundles in a way that respects the C^r structure.

The main tool in proving principal bundle superrigidity is an object which, in the book at hand, is termed an "H-pair". Perhaps H-pairs are present in the background of Zimmer's proof of cocycle superrigidity in [Z84], and in Margulis' work, as well. Here they are given a name, they are a central object of interest and some effort is spent on their development. Once the basic yoga for manipulating H-pairs is established, the principal bundle superrigidity theorem follows from a wonderfully short and painless argument.

The remainder of the book (up to the first appendix) is a denouement which is devoted to applications of principal bundle superrigidity. In Section 10.6, Feres shows that principal bundle superrigidity implies archimedean cocycle superrigidity. Since Zimmer's Cocycle Superrigidity Theorem generalizes Margulis' Superrigidity Theorem, one also sees that archimedean superrigidity for lattices is a consequence of principal bundle superrigidity. Section 10.7 describes how cocycle superrigidity gives information about Lyapunov spectrum, for actions of semisimple Lie groups of real rank ≥ 2.

The dynamical superrigidity theorems described above are all for actions of connected semisimple Lie groups (of real rank ≥ 2); there are no lattices. However, there is a process called "induction", which can take an action of a lattice and produce an action of the ambient Lie group. This process is often described only for actions on manifolds, but it also works equally well for actions on principal bundles. Consequently, to study lattice actions on principal bundles, one may induce up to an action of the ambient group and apply principal bundle superrigidity to this. This technique is described in the final section of the book, Section 10.8.

2c. **Further remarks.** I came across only three or four typographical errors in my reading, and they were all obvious and easily fixed, so I will not bother to list them. More probably exist, but I estimate that they should be no problem to the interested reader.

The index is good, but not excellent. One does not find, for example, the Borel Density Theorem, Oseledec' Theorem, the ergodic theorem or Rosenlicht's Theorem. I did not keep a more extensive list, but it did happen to me a few times, while writing this review, that I had a little difficulty finding my way to a specific result.

The goal of the book at hand is principal bundle superrigidity, and it does not stop for discursive asides. For example, I found no comment about M. Ratner's work on actions on homogeneous spaces by unipotent-generated groups. (For an overview of this theory, one may consult [R95].)

The only material I saw that was related to arithmeticity of lattices appears in §A.2 of Appendix A. However, even there, the intent is to expose A. Borel's theorem on existence of cocompact lattices and not Margulis' Arithmeticity Theorem. As

I think about the subject, superrigidity and arithmeticity are closely intertwined subjects and a discussion of one without the other leaves me with the feeling of a missing tooth.

The program of Mackey and Zimmer has been unsuccessful in finding a good dynamical form of arithmeticity, so the world of dynamics is poorer than the world of lattices. It is understandable, then, that arithmeticity would not be under discussion in this book, but the concern I have is that the Superrigidity Theorem itself may appear to the novice as a cumbersome technical result that is hard to apply and which comes from nowhere. This is most unfortunate, since superrigidity/arithmeticity is one of the greatest achievements of 20th century mathematics. On the other hand, this is a problem to which I claim no solution.

From my own experience, I can assert that it is very difficult to discuss superrigidity/arithmeticity with a non-expert in a way that is accurate and that, at the same time, makes clear the fundamental importance and utility of the theory.

2d. **Conclusion.** The book under review is well written and is highly recommended. It will be a great help to the next generation of mathematicians interested in dynamical superrigidity.

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