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Arithmeticity in the theory of automorphic forms, by Goro Shimura, Mathematical Surveys and Monographs, vol. 82, American Mathematical Society, Providence, RI, x+302 pp., $69.00, ISBN 0-8218-2671-9

This book is a companion to the author's previous book [11], *Euler products and Eisenstein series*, published by the AMS. The books' ultimate objective is to prove algebraicity of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups. In the course of the study of the zeta functions, many important results, which were obtained by the author during 1960-2000, are exposed.

In fact, the first four chapters form a very nice textbook addressed to advanced graduate students and researchers, while the latter three chapters describe the author's recent research on critical values, which was never in print. It is well known that Shimura's mathematics developed by stages: (A) Complex multiplication of abelian varieties $\Longrightarrow$ (B) The theory of canonical models = Shimura varieties $\Longrightarrow$ (C) Critical values of zeta functions and periods of automorphic forms. (B) includes (A) as the 0-dimensional special case of canonical models. The relation of (B) and (C) is more involved, but (B) provides a solid foundation of the notion of the arithmetic automorphic forms. In some sense, this book can be regarded as the culmination of the author's research.

The study of critical values is indispensable to developing p-adic theory. Also unitary Shimura varieties have recently attracted the interest of increasingly many researchers producing many applications. In this regard, the publication of this book is timely and will be welcomed by the mathematical public.

1. Short history on critical values until 1975

In 1735, Euler discovered experimentally $\zeta(2) = \pi^2/6$ and proved $\zeta(2n)/\pi^{2n} \in \mathbf{Q}$ expressing the quotient by the n-th Bernoulli number for $1 \leqq n \in \mathbf{Z}$ in 1742 ([13], p. 261, p. 272). Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. This was the first serious result on the critical values of the zeta function.[1] In 1899, Hurwitz showed [2] in analogy to Euler's theorem that $(\sum_z z^{-4m})/\varpi^{4m} \in \mathbf{Q}$ for $1 \leqq m \in \mathbf{Z}$, where $z = a + bi$, a, $b \in \mathbf{Z}$ extends over all non-zero Gaussian integers and $\varpi = 2\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$. In the 19th century, the interest of great number theorists in the zeta functions was rather confined to those associated to number fields, influenced by the prototypes of Dirichlet-Riemann. Early in the 20th century, Hecke introduced the L-function associated with a modular form besides the L-function associated with a Grössencharacter (now called a Hecke character). The

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[1]The notion of critical values, which is generally accepted now, can be defined as follows. Suppose that a zeta function $Z(s)$ multiplied by its gamma factor $G(s)$ satisfies a functional equation of standard type under the symmetry $s \to v - s$. Then $Z(n)$, $n \in \mathbf{Z}$ is a critical value of $Z(s)$ if both $G(n)$ and $G(v-n)$ are finite. The older result of Leibnitz, $1-1/3+1/5-1/7+\cdots = \pi/4$, can be regarded as a result on a critical value.

simplest such L-function is $L(s,\Delta)=\sum_{n=1}^{\infty}a_n n^{-s}$ where

$$\Delta(z)=\sum_{n=1}^{\infty}a_n q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \qquad q=\exp(2\pi i z)$$

is the cusp form of weight 12 with respect to $SL(2,\mathbf{Z})$.

In 1959, Shimura [6] showed, as an application of Eichler-Shimura theory on period integrals and cohomologies of Fuchsian groups, that

$$\pi^m L(m,\Delta)/\pi^n L(n,\Delta)\in\mathbf{Q} \quad \text{if} \quad m\equiv n \mod 2,\ 1\leqq m,n\leqq 11,$$

and calculated the quotients explicitly.[2] We can write this result as

$$\frac{L(n,\Delta)}{(2\pi i)^n c^{\pm}(\Delta)}\in\mathbf{Q}, \qquad 1\leqq n\leqq 11, \quad \pm 1=(-1)^n \tag{1}$$

with "periods" $c^{\pm}(\Delta)$ of Δ. Shimura's method of performing this calculation was later developed into the theory of modular symbols by Manin [5].

2. Critical values and an integral representation

In 1976, Shimura [9] discovered a completely new method: how to prove the algebraicity of critical values when an integral representation of an L-function is given. Since this paper contains basic ideas in their primitive forms, which are developed fully in this book, it is appropriate to explain them briefly here.

For a positive integer N, we define a congruence subgroup $\Gamma_0(N)$ of $SL(2,\mathbf{Z})$ by $\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbf{Z})\,\middle|\, c\equiv 0 \mod N\right\}$. Let $\mathfrak{H}$ denote the complex upper half plane. For $0\leqq\lambda\in\mathbf{Z}$, $z=x+iy\in\mathfrak{H}$ and $s\in\mathbf{C}$, we define an Eisenstein series $E_\lambda(z,s)$ by

$$E_\lambda(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}(cz+d)^{-\lambda}|cz+d|^{-2s}, \qquad \Gamma_\infty=\left\{\pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\,\middle|\, m\in\mathbf{Z}\right\},$$

where $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ extends over a complete set of representatives of $\Gamma_\infty\backslash\Gamma_0(N)$. The sum converges when $\Re(2s)>2-\lambda$. The analytic behavior of $E_\lambda(z,s)$ is well known. It can be continued to a meromorphic function on the whole s-plane. If $\lambda>0$, it is a holomorphic function of s on $\Re(s)\geqq 1/2-\lambda/2$.

Define a differential operator and its iterations by

$$\delta_\lambda=\frac{1}{2\pi i}\left(\frac{\partial}{\partial z}+\frac{\lambda}{2iy}\right), \quad y=\Im(z), \qquad \delta_\lambda^{(r)}=\delta_{\lambda+2r-2}\cdots\delta_{\lambda+2}\delta_\lambda, \qquad \delta_\lambda^{(0)}=1.$$

δ_λ is the differential operator which sends C^∞-automorphic forms of weight λ to those of weight $\lambda+2$. Assuming $\lambda>0$, we put $E_\lambda(z)=E_\lambda(z,0)$. Then it can be shown that

$$E_{\lambda+2r}(z,-r)=\frac{\Gamma(\lambda)}{\Gamma(\lambda+r)}(-4\pi y)^r\delta_\lambda^{(r)}E_\lambda(z). \tag{2}$$

Let $f=\sum_{n=1}^{\infty}a_n q^n$, $g=\sum_{n=0}^{\infty}b_n q^n$ be modular forms with respect to $\Gamma_0(N)$ of weights k and l respectively.[3] Define a Dirichlet series by $D(s,f,g)=$

[2] Strictly speaking, the information for $L(11,\Delta)$ is missing in [6]. But this can be remedied easily using the same method by passing to a congruence subgroup $\Gamma_0(2)$.

[3] For simplicity, we describe the results for modular forms with the trivial character. We also assume that $a_n\in\mathbf{R}$.

$\sum_{n=1}^{\infty} a_n b_n n^{-s}$. We assume that $k > l$ and f is a cusp form. By the Rankin-Selberg method, we have

$$(3) \qquad (4\pi)^{-s}\Gamma(s)D(s,f,g) = \int_{\Gamma_0(N)\backslash\mathfrak{H}} \overline{f(z)}g(z)E_{k-l}(z, s+1-k)y^{s-1}dxdy.$$

Take $r \in \mathbf{Z}$ so that $0 \leqq r$, $l + 2r < k$. In (3), we put $s = k - 1 - r$ and using (2) with $\lambda = k - l - 2r$, we obtain

$$(4) \qquad D(k-1-r, f, g) = c\pi^k \langle f, g\delta_\lambda^{(r)} E_\lambda(z)\rangle, \qquad c \in \mathbf{Q}^\times.$$

Here $\langle\ ,\ \rangle$ denotes the normalized Petersson inner product; i.e., we put

$$\langle h_1, h_2\rangle = \mathrm{vol}(\Gamma_0(N)\backslash\mathfrak{H})^{-1} \int_{\Gamma_0(N)\backslash\mathfrak{H}} \overline{h_1(z)}h_2(z)y^{k-2}dxdy$$

for C^∞-automorphic forms of weight k with respect to $\Gamma_0(N)$ whenever the integral is convergent.

In (4), $g\delta_\lambda^{(r)}E_\lambda(z)$ is an automorphic form of weight k which is not holomorphic in general. However its nonholomorphic character can be controlled by the concept of near holomorphy. The point is that such forms can be decomposed as

$$(5) \qquad g\delta_\lambda^{(r)}E_\lambda(z) = \sum_{m=0}^{t} \delta_{k-2m}^{(m)} h_m, \qquad t = r \text{ or } r+1$$

with a holomorphic modular form h_m of weight $k - 2m$ with respect to $\Gamma_0(N)$ and that

$$(6) \qquad \langle h, g\delta_\lambda^{(r)}E_\lambda(z)\rangle = \langle h, h_0\rangle$$

for every holomorphic cusp form h of weight k. Therefore (5) gives the orthogonal projection to the space of holomorphic cusp forms of weight k.

Another important aspect is the compatibilities with the action of the automorphism group of $\mathbf{C}$, which we denote by $\mathrm{Aut}(\mathbf{C})$. We define the action of $\mathrm{Aut}(\mathbf{C})$ on a modular form through its Fourier coefficients. Thus if f is as above, $f^\sigma = \sum_{n=1}^{\infty} a_n^\sigma q^n$ and f^σ is a modular form of weight k with respect to $\Gamma_0(N)$. Then the decomposition (5) can be done in a compatible way with the action of $\mathrm{Aut}(\mathbf{C})$. Now assume that f is a normalized Hecke eigenform. Then

$$(7) \qquad \left(\frac{\langle f, h\rangle}{\langle f, f\rangle}\right)^\sigma = \frac{\langle f^\sigma, h^\sigma\rangle}{\langle f^\sigma, f^\sigma\rangle}, \qquad \sigma \in \mathrm{Aut}(\mathbf{C})$$

holds for any holomorphic modular form h. Now it follows from (4) that

$$(8) \qquad \left(\frac{D(k-1-r, f, g)}{\pi^k\langle f, f\rangle}\right)^\sigma = \frac{D(k-1-r, f^\sigma, g^\sigma)}{\pi^k\langle f^\sigma, f^\sigma\rangle}, \qquad 0 \leqq r < (k-l)/2$$

for $\sigma \in \mathrm{Aut}(\mathbf{C})$. The theorem similar to (1) on critical values of the L-function $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$ can be obtained from (8) taking a suitable Eisenstein series as g. This method gives more information about the periods of modular forms; we have $c^+(\Delta)c^-(\Delta) = i\pi\langle\Delta, \Delta\rangle$ for example.

This sketch shows that the main ingredients to deduce arithmetical information from an integral representation are: (i) Arithmeticity of automorphic forms; i.e., at least we must be able to define $\overline{\mathbf{Q}}$-rational automorphic forms. (ii) An explicit integral representation of the L-function to be studied. (iii) Good knowledge of Eisenstein series involved in (ii). (iv) Differential operators, the notion of near holomorphy and a suitable expression of the orthogonal projection.

Paper [9] has been extremely influential. Here we mention only Sturm [12] and Garrett [1] among numerous other papers.

3. The groups and automorphic forms studied in this book

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. We define

$G = Sp(n, F)$ (Case Sp),

$G = \{\alpha \in GL_{2n}(K) \mid \alpha\eta_n\alpha^* = \eta_n\}$, $\eta_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$ (Case UT=unitary tube),

$G = \{\alpha \in GL_n(K) \mid \alpha T\alpha^* = T\}$, (Case UB = unitary ball),

according to three cases. Here $\alpha^* = {}^t\alpha^\rho$, $T \in GL_n(K)$, $T^* = -T$.

Assume that $F = \mathbf{Q}$ for a while. The group of the real points G_∞ acts on the associated symmetric domain

$$(9) \qquad H = \begin{cases} \{z \in M(n, n, \mathbf{C}) \mid {}^tz = z,\ \Im(z) > 0\} & \text{(Case Sp)}, \\ \{z \in M(n, n, \mathbf{C}) \mid i(z^* - z) > 0\} & \text{(Case UT)}, \\ \{z \in M(p, q, \mathbf{C}) \mid 1_q - z^*z > 0\} & \text{(Case UB)}, \end{cases}$$

(p, q), $p + q = n$ being the signature of iT. Here $z^* = {}^t\bar{z}$ and > 0 means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g, z)$, $g \in G_\infty$, $z \in H$ taking values in $GL_p(\mathbf{C}) \times GL_q(\mathbf{C})$. Let ω be a representation of this group on a finite dimensional vector space X over $\mathbf{C}$. Let Γ be a congruence subgroup of G. We call a holomorphic function f on H taking values in X *a holomorphic automorphic form of weight* ω (with respect to Γ) if

$$f(\gamma z) = \omega(M(\gamma, z))f(z) \qquad \text{for all} \quad \gamma \in \Gamma,\ z \in H$$

is satisfied. (When $n = 2$, cusp conditions are required.) We can define C^∞ and meromorphic automorphic forms similarly; Cases SP and UT are similar.

To deal with general F, it is natural to consider automorphic forms on G_A, where G_A is the adele group of G over F. The infinite part G_∞ of G_A acts on the symmetric domain H which is the product of domains of the form (9). An automorphic form on G_A is essentially a suitable collection of automorphic forms on H of the same weight. In Case Sp, automorphic forms of half integral weight and their adelic versions as automorphic forms on the metaplectic group are also introduced.

Cases Sp and UT have a simpler feature that H is a tube domain, which implies that the automorphic forms have Fourier expansions of standard type:

$$f(z) = \sum_h a_h \exp(2\pi i\,\mathrm{tr}(hz)), \qquad z \in H, \quad a_h \in X.$$

This is the reason why the case UT is treated separately and plays an important role, though it is a special case of UB.

4. Arithmeticity of automorphic forms

A background of the arithmeticity is the theory of canonical models. In a weak form, the theory goes as follows. For a congruence subgroup Γ of G, $\Gamma\backslash H$ has the structure of a Zariski open subset of a projective algebraic variety (Bailey-Borel-Satake compactification). The theory says that $\Gamma\backslash H$ has a model over a specified algebraic number field k_Γ; thus we have the notion of automorphic functions rational over k_Γ; furthermore the value of such an automorphic function at a CM-point[4] on H is algebraic (if it is finite) and satisfies the explicit reciprocity law. Since CM-points are dense in H, these properties determine the model uniquely. In this book, an almost self-contained new proof of the existence of canonical models for G is given.

We may assume that X has a $\overline{\mathbf{Q}}$-structure, i.e., $X = X_0 \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$, with a vector space X_0 over $\overline{\mathbf{Q}}$. Then the holomorphic arithmetic ($\overline{\mathbf{Q}}$-rational) automorphic forms are defined by requiring their Fourier coefficients to be in X_0 in Cases Sp and UT. The definition can be extended to meromorphic forms. This notion satisfies a number of compatibility constraints.

1) In the case where the weight is 0 (i.e., the case ω is trivial), the arithmeticity coincides with the rationality over $\overline{\mathbf{Q}}$ in the theory of canonical models. If f and g are scalar valued arithmetic automorphic forms of the same weight, f/g is an arithmetic automorphic function. It takes algebraic values at all CM-points where it is finite.

2) If f is an arithmetic automorphic form and w is a CM-point on H at which f is finite, then $\omega(\mathfrak{p}(w))^{-1} f(w) \in X_0$, where $\mathfrak{p}(w)$ is a certain matrix whose entries can be written explicitly by CM-periods.[5]

3) The space of arithmetic automorphic forms is stable under the action of G; it spans the space of automorphic forms over $\mathbf{C}$.

For example, we consider an Eisenstein series

$$E_k(z) = \sum_{(c,d)}{}' (cz+d)^{-k}, \qquad z \in \mathfrak{H},$$

for $4 \leqq k \in \mathbf{Z}$, where the summation extends over all pairs of integers (c, d) excluding $(0, 0)$. There is the relation $E_k(z) = 2\zeta(2k)E_k(z, 0)$ with the Eisenstein series $E_\lambda(z, s)$ in §2 taking $N = 1$ there. It is well known that $E_k(z)$ is a holomorphic modular form of weight k and that $\pi^{-k}E_k(z)$ has rational Fourier coefficients. Since $\mathfrak{p}(w) = \pi^{-1}\varpi$ for the CM-point $w = i$, we have $\varpi^{-k}E_k(i) \in \overline{\mathbf{Q}}$ from 2), which is consistent with Hurwitz's theorem mentioned in §1.

The explicit reciprocity law is formulated as an action of $\mathcal{G}_+ \times \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the space of meromorphic arithmetic automorphic forms, where $G_A \subset \mathcal{G}_+$ is a certain subgroup $\widetilde{G}_A$, $\widetilde{G}$ being the similitude group. This theorem is very powerful, containing all known reciprocity laws in the theory of complex multiplication.

In the case UB, the Fourier expansion is not available in general, so 2) is adapted as the definition of the arithmeticity; 1) and 3) still hold.

5. Differential operators and near holomorphy

Various kinds of differential operators are introduced. Typical operators are of the following type. For simplicity assume $F = \mathbf{Q}$. Let K^c be the complexified form of the maximal compact subgroup of G_∞. Let T be the tangent space of H at a point. K^c acts on T. Let $S_p(T)$ be the vector space of all homogeneous polynomial functions on T of degree p. For an irreducible subspace Z of $S_p(T)$, a

[4]The common fixed point on H of a maximal torus of G which is anisotropic over $\mathbf{R}$ is called a CM-point.

[5]By a CM-period, we mean a monomial of periods of abelian varieties with complex multiplication.

differential operator D^Z_ω is defined; D^Z_ω sends C^∞-automorphic forms of weight ω to C^∞-automorphic forms of weight $\omega \otimes Z$, identifying Z with the representation which it defines.

For example, if $G = Sp(n, \mathbf{Q})$, then

$$T = \{z \in M(n, n, \mathbf{C}) \mid {}^t z = z\}, \qquad K^c = GL(n, \mathbf{C}).$$

$K^c \ni a$ acts on T by $z \to az{}^t a$. Clearly the representation $x \to \det(x)^2$ occurs in $S_n(T)$ with multiplicity 1. For the corresponding subspace Z, D^Z_ω gives, for $\omega(x) = (\det x)^k$, the operator raising the weight by 2. When $n = 1$, this is $2\pi i$ times the operator δ_k in §2. If the contragredient action $z \to {}^t a^{-1} z a^{-1}$ is taken, we get the operator lowering the weight by 2.

The K-types of representations of a semi-simple Lie group are well investigated (cf. [3], Chapter XV, for example). The elements in the universal enveloping algebra which send one K-type vector to another one could be explicitly written in principle, but the task will be combinatorially formidable. The operators studied in this book can be regarded as the universal operators and will be practical for many purposes.

Next the concept of near holomorphy is introduced for a complex Kähler manifold V. When $V = H$ in Cases Sp and UT, nearly holomorphic functions are polynomials of $r_1, \dots, r_m$ with holomorphic functions on H as coefficients; here r_i are entries of $({}^t z - \bar{z})^{-1}$, $z \in H$. The arithmeticity of nearly holomorphic automorphic forms can be defined so that it satisfies the compatibilities 2), 3) and the last statement of 1) of §4. The differential operators of the form $\pi^{-e} D^Z_\omega$, $0 \leqq e \in \mathbf{Z}$ preserve near holomophy and arithmeticity.

6. Integral representations and critical values of the zeta functions

In this part, automorphic forms are assumed to be scalar valued. First, for Cases Sp and UT, Eisenstein series $E(z, s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behavior and those $\sigma \in 2^{-1}\mathbf{Z}$ at which $E(z, \sigma)$ is nearly holomorphic and arithmetic are investigated. This is achieved by proving a relation similar to (2) involving a differential operator and examining Fourier coefficients of Eisenstein series using the theory of hypergeometric functions on tube domains [10].

For a Hecke eigenform $\mathbf{f}$ on G_A and an algebraic Hecke character χ of the idele group of K (we put $K = F$ in Case Sp), the zeta function $Z(s, \mathbf{f}, \chi)$ is defined.[6] Regarded as an Euler product extending over prime ideals of F, the degree of its Euler factor is $2n + 1$ in Case Sp, $4n$ in Case UT and $2n$ in Case UB, except for finitely many prime ideals. This zeta function is almost the same as the so called standard L-function attached to $\mathbf{f}$ twisted by χ, but actually is more general than that in the unitary case.

Let us consider Case UB and explain the main ideas in the simplest case. In the previous book [11], the integral representation of the following type is established. Let T be the skew hermitian matrix of size n as in §3 and write the group there as $G = U(T)$. Let $\mathcal{H}^T$ denote the associated symmetric domain. For $q \geqq 0$, put $T_q = T \oplus \eta_q$, which is a matrix of size $n + 2q$; T and η_q are placed on the diagonal blocks. $U(T_q)$ has a parabolic subgroup whose Levi part is isomorphic to

[6] The analytic part of the theory is developed without assuming the algebraicity of χ.

$U(T) \times GL_q(K)$. For a cusp form $\mathbf{f}$ on $U(T)_A$ (we regard it as a function on $\mathcal{H}^T$) and a Hecke character χ as above, the Eisenstein series $E(z,s;\mathbf{f},\chi)$ associated to $(\mathbf{f},\chi)$ is introduced. This function defined on $\mathcal{H}^{T_q}$ is similar to "Eisenstein series associated to a cusp form" introduced by Langlands [4] and Klingen. Put $\widetilde{T} = T_q \oplus (-T)$, $H_{n+q} = \mathcal{H}^{\eta_{n+q}}$. Then $U(T_q) \times U(T) \subset U(\widetilde{T}) \cong U(\eta_{n+q})$, and we have the induced embedding of the symmetric domain $\mathcal{H}^{T_q} \times \mathcal{H}^T \to H_{n+q}$. Let $H(z,w;s)$, $z \in \mathcal{H}^{T_q}$, $w \in \mathcal{H}^T$ be the pull back (the restriction) of the Eisenstein series $E(\tilde{z},s)$ of Siegel type on H_{n+q}, which has been investigated already in detail. Then it is proved that

$$c(s)Z(s,\mathbf{f},\chi)E(z,s;\mathbf{f},\chi) = \Lambda(s)\int_{\Gamma\backslash\mathcal{H}^T} H(z,w;s)\mathbf{f}(w)\delta(w)^k dw. \tag{10}$$

Here $c(s)$ is an explicitly given gamma factor and $\Lambda(s)$ is a product of L-functions of F; k is the weight of $\mathbf{f}$ and $\delta(w)$ is the determinant of the imaginary part of w. When $q = 0$, (10) takes the form

$$c(s)Z(s,\mathbf{f},\chi)\mathbf{f}(z) = \Lambda(s)\int_{\Gamma\backslash\mathcal{H}^T} H(z,w;s)\mathbf{f}(w)\delta(w)^k dw. \tag{11}$$

Now for $\sigma \in 2^{-1}\mathbf{Z}$ in a certain finite interval, $E(\tilde{z},\sigma)$ is nearly holomorphic and arithmetic. Then $H(z,w;\sigma)$ is a finite linear combination of the products of arithmetic nearly holomorphic functions on $\mathcal{H}^T$ multiplied by an explicit period invariant. In this way, a main result is derived: It is

$$Z(\sigma,\mathbf{f},\chi) \in \pi^A P\langle\mathbf{f},\mathbf{f}\rangle\overline{\mathbf{Q}}, \qquad A \in \mathbf{Z} \tag{12}$$

for critical values of the zeta function, where P is a CM-period and $\langle\mathbf{f},\mathbf{f}\rangle$ is the normalized Petersson inner product. (The reason that half integers appear in critical values is that a suitable shift of the variable s is made in the definition of the zeta function.) Next, applying this result together with a nonvanishing theorem on the zeta values to (10), we establish the arithmeticity and the near holomorphy of $E(z,\sigma;\mathbf{f},\chi)$.

Similar results hold for Cases SP and UT with $P = 1$. A self-contained proof of the fundamental formula (10) for these cases is given in this book. It is worth noting that another proof of (12) is given for Cases Sp and UT. It is based on an integral representation of $\mathcal{Z}(s,\mathbf{f},\chi)$ of the type of the Rankin-Selberg convolution of $\mathbf{f}$ with a theta function, similar to the method in [8].

Also noteworthy is that parallel results are obtained for the zeta function of an automorphic form of half integral weight in Case Sp. The degree of its Euler factor is $2n$. It may be expected that these zeta functions are related to those of automorphic forms on $O(2n+1)$ and also to the zeta functions of n-dimensional abelian varieties.

7. Conclusion

This book is written in a scholarly and careful manner to bring the reader to the forefront of the research on critical values of zeta functions. Though ambitious in its purpose, the prerequisites are not huge: basic knowledge of algebra, algebraic number theory, complex analysis, algebraic geometry on an elementary level and abelian varieties over $\mathbf{C}$. Beside these, the reading of the first three chapters of [7] and [9] will be helpful.

In the appendix, several basic facts are proved to make the book self-contained: theta series, metaplectic groups, etc. The results on modules over a universal enveloping algebra (generated by automorphic forms) are especially interesting and new.

Anyone who seriously wants to study critical values of zeta functions should keep this book near his/her desk.

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